

STRUCTURE OF CLASSICAL AFFINE AND CLASSICAL AFFINE FRACTIONAL \mathcal{W} -ALGEBRAS

UHI RINN SUH

ABSTRACT. We show that one can construct a classical affine \mathcal{W} -algebra via a classical BRST complex. This definition clarifies that classical affine \mathcal{W} -algebras can be considered as quasi-classical limits of quantum affine \mathcal{W} -algebras.

We also give a definition of a classical affine fractional \mathcal{W} -algebra as a Poisson vertex algebra. As in the classical affine case, a classical affine fractional \mathcal{W} -algebra has two compatible λ -brackets and is isomorphic to an algebra of differential polynomials as a differential algebra. When a classical affine fractional \mathcal{W} -algebra is associated to a minimal nilpotent, we describe explicit forms of free generators and compute λ -brackets between them. Provided some assumptions on a classical affine fractional \mathcal{W} -algebra, we find an infinite sequence of integrable systems related to the algebra, using the generalized Drinfel'd and Sokolov reduction.

CONTENTS

1. Introduction	2
2. Poisson vertex algebras and Integrable systems	3
2.1. Lie conformal superalgebras and Poisson vertex algebras	3
2.2. Integrable systems	6
3. Two equivalent definitions of classical affine \mathcal{W} -algebras	9
3.1. The first definition of classical affine \mathcal{W} -algebras	10
3.2. The second definition of classical affine \mathcal{W} -algebras	12
3.3. Equivalence of the two definitions of classical affine \mathcal{W} -algebras	13
4. Two equivalent definitions of classical affine fractional \mathcal{W} -algebras	17
4.1. First definition of classical affine fractional \mathcal{W} -algebras	19
4.2. Second definition of classical affine fractional \mathcal{W} -algebras	24
5. Generating elements of a classical affine fractional \mathcal{W} -algebra and Poisson λ -brackets between them	28
5.1. Generating elements of classical affine fractional \mathcal{W} -algebras	28
5.2. Examples	29
5.3. More results on generating elements of a classical affine fractional \mathcal{W} -algebra associated to a minimal nilpotent	32
6. Integrable systems related to classical affine fractional \mathcal{W} -algebras	39
References	45

1. INTRODUCTION

An affine classical \mathcal{W} -algebra is closely related to the theory of integrable systems. The connection between two structures was constructed by Fateev and Lukyanov [6]. Moreover, Drinfel'd and Sokolov [13] explained that, given a Lie algebra and a principle element, an affine classical \mathcal{W} -algebra is obtained by the Drin'feld-Sokolov Hamiltonian reduction.

To be precise, let \mathfrak{g} be a simple Lie algebra, let (e, h, f) be an \mathfrak{sl}_2 -triple with a principle nilpotent f and let $\mathfrak{g}(i)$ be the eigenspace $\{g \in \mathfrak{g} | [\frac{h}{2}, g] = ig\}$. Consider the Lax operator

$$(1.1) \quad L = \partial_x + q(x, t) + \Lambda, \quad x \in S^1, \quad t \in \mathbb{R},$$

where $\Lambda = -f - pz^{-1} \in \mathfrak{g}[z, z^{-1}]$, $p \in \mathfrak{g}$ is a central element of $\mathfrak{n} = \bigoplus_{i \geq 0} \mathfrak{g}(i)$ and $q(x, t) \in \bigoplus_{i > -1} \mathfrak{g}(i)$. The gauge transformation by $S \in \mathfrak{n}$ on the space of matrix valued smooth functions, $S^1 \rightarrow \bigoplus_{i > -1} \mathfrak{g}(i)$, is defined by $q(x, t) \mapsto \tilde{q}(x, t)$, where $e^{\text{ad} S} L = \partial_x + \tilde{q}(x, t) + \Lambda$. Then the gauge invariant functionals consist of the affine classical \mathcal{W} -algebra associated to \mathfrak{g} and f . Moreover, the \mathcal{W} -algebra is associated to an integrable system obtained by a commutator of the Lax operator.

Burroughs, De Groot, Hollywood and Miramontes generalized this idea replacing Λ by $\Lambda_m = z^{-m} \cdot \Lambda \in \mathfrak{g}[z, z^{-1}]$ and $q(x)$ by $q_m(x) \in \bigoplus_{j=-m+1}^0 \mathfrak{g}z^j \oplus \bigoplus_{i > -1} \mathfrak{g}(i)z^{-m}$ (see [1, 3]). In this way, a classical affine fractional \mathcal{W} -algebra associated to \mathfrak{g} and Λ_m was described as a differential algebra using gauge invariant functionals. The authors also explained relations between fractional \mathcal{W} -algebras and integrable systems.

On the other hand, Barakat, De Sole and Kac [2] developed the theory of Hamiltonian equations in the language of the Poisson vertex algebra (PVA) theory. In particular, the authors described the Drinfel'd-Sokolov Hamiltonian reduction using λ -brackets and defined a classical affine \mathcal{W} -algebra as a PVA. Indeed, the \mathcal{W} -algebra in Drinfel'd-Sokolov [13] and the \mathcal{W} -algebra in Barakat-De Sole-Kac [2] are equivalent as differential algebras (see [2, 5]).

As one predicts from the name of a ‘‘classical affine’’ \mathcal{W} -algebra, there are other three types of \mathcal{W} -algebras: classical finite, quantum finite and quantum affine \mathcal{W} -algebras. These three types of \mathcal{W} -algebras have algebraic structures of Poisson algebra, Lie algebra and Vertex algebra, respectively. In [8], Gan and Ginzburg introduced two equivalent definitions of a quantum (classical) finite \mathcal{W} -algebra, by a Lie algebra cohomology and by a Hamiltonian reduction. In the quantum affine case, a \mathcal{W} -algebra is defined by a BRST-complex, which can be considered as a quantization of the Lie algebra cohomology in [8] (see [4]). It is still open that if a quantum affine \mathcal{W} -algebra has an analogous definition to the definition of a finite \mathcal{W} -algebra by a Hamiltonian reduction.

In this paper, we give a new construction of a classical affine \mathcal{W} -algebra by a classical BRST-complex and we prove that the two definitions, via a classical BRST-complex and via a Hamiltonian reduction, are equivalent. Moreover, we define a classical affine fractional \mathcal{W} -algebra as a PVA with two compatible λ -brackets.

Another main result of this paper is finding free generators of classical affine \mathcal{W} -algebras, as differential algebras. In general, finding free generators of \mathcal{W} -algebras is not easy. However, provided that f is a minimal nilpotent in \mathfrak{g} , Premet found the generating elements of finite quantum \mathcal{W} -algebras associated to f as associative algebras. Also, they computed

the Lie brackets between the generators. In the affine quantum cases, Kac and Wakimoto found the generating elements of \mathcal{W} -algebras associated to f as differential algebras and they computed λ -brackets between the generators. (See [10], [11] and [12].)

As in the other cases, with the assumption that f is a minimal nilpotent, one can find free generators of affine classical \mathcal{W} -algebras and Poisson λ -brackets between them [16]. Moreover, in this paper, we describe generating elements of classical affine fractional \mathcal{W} -algebras and λ -brackets between them.

Outline of this paper

In Section 2, we review several notions which are used in following sections. In Section 2.1, (nonlinear) Lie conformal algebras and Poisson vertex algebras are introduced, and in Section 2.2 integrable systems are explained in the theory of PVAs.

In Section 3.1 and 3.2, we give a definition of a classical affine \mathcal{W} -algebra by a classical BRST complex and by a Hamiltonian reduction, respectively. In Section 3.3, we prove the two definitions in Section 3.1 and Section 3.2 are equivalent, which is the first goal of this paper.

In Section 4, we derive a definition of a classical affine fractional \mathcal{W} -algebra as a PVA. In Section 4.1 we review the construction of a classical affine fractional \mathcal{W} -algebra explained in [1, 3] and show this algebra has two local Poisson brackets. In Section 4.2, we give PVA structures on the algebra using relations between local Poisson brackets and λ -brackets. The definition of a classical affine fractional \mathcal{W} -algebra in Section 4.2 is analogous to the definition of a classical affine \mathcal{W} -algebra via Hamiltonian reduction.

In Section 5, we find generating elements of classical affine fractional \mathcal{W} -algebras when the algebras are associated to semisimple elements. In Section 5.1 and 5.2, we explain how to find generators of classical affine fractional \mathcal{W} -algebras and we give an example using the method. In Section 5.3, we describe explicit formulas of generating elements of a classical affine fractional \mathcal{W} -algebra when it is associated to a minimal nilpotent element. We notice that all the results in this section hold for classical affine \mathcal{W} -algebras as corollaries and these are written in [16].

In Section 6, we explain relations between integrable systems and classical affine fractional \mathcal{W} -algebras using the language of PVAs. Here, the idea comes from [1, 3].

Acknowledgement

I would like to thank my Ph.D. thesis advisor, Victor Kac, for valuable discussions.

2. POISSON VERTEX ALGEBRAS AND INTEGRABLE SYSTEMS

2.1. Lie conformal superalgebras and Poisson vertex algebras. An associative algebra \mathcal{D} endowed with a linear operator $\partial : \mathcal{D} \rightarrow \mathcal{D}$ is called a differential algebra if the operator ∂ satisfies

$$\partial(AB) = A\partial(B) + \partial(A)B, \text{ for } A, B \in \mathcal{D}.$$

Let $I = \{1, \dots, l\}$ be an index set. An important example of differential algebras is the algebra of differential polynomials $\mathbb{C}_{\text{diff}}[a_i \mid i \in I] := \mathbb{C}[a_i^{(n)} \mid a_i^{(n)} := \partial^n a_i, i \in I, n \in \mathbb{Z}_{\geq 0}]$ in the variables a_1, \dots, a_l .

Now we introduce a Lie conformal superalgebra and a Poisson vertex algebra. For this purpose, we review $\mathbb{Z}/2\mathbb{Z}$ -graded algebraic structures and a λ -bracket on a $\mathbb{C}[\partial]$ -module.

- Definition 2.1.** (i) A vector superspace V is a vector space with a $\mathbb{Z}/2\mathbb{Z}$ -graded decomposition $V = V_{\bar{0}} \oplus V_{\bar{1}}$. We call $V_{\bar{0}}$ the even space and $V_{\bar{1}}$ the odd space.
(ii) Let V be a vector superspace and $a \in V_{\bar{i}}$. Then we say parity p of a is i and we write $p(a) = i$.
(iii) Given a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$, the algebra $\text{End } V$ acquires a $\mathbb{Z}/2\mathbb{Z}$ -grading by letting

$$(\text{End } V)_{\bar{\alpha}} = \{A \in \text{End } V \mid A(V_{\bar{\beta}}) \subset V_{\bar{\alpha}+\bar{\beta}}\}$$

for $\bar{\alpha}, \bar{\beta} \in \mathbb{Z}/2\mathbb{Z}$.

- (iv) A commutative superalgebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is a superalgebra such that $ab = p(a, b)ba$, where $a, b \in A_{\bar{0}} \cup A_{\bar{1}}$ and $p(a, b) = (-1)^{p(a)p(b)}$.

Definition 2.2 (λ -bracket). Let R be a $\mathbb{C}[\partial]$ -module. A λ -bracket $[\cdot, \cdot]_{\lambda} : R \otimes R \rightarrow R[\lambda]$ on R is a \mathbb{C} -bilinear map satisfying

$$(2.1) \quad [\partial a_{\lambda} b] = -\lambda[a_{\lambda} b] \text{ and } [a_{\lambda} \partial b] = (\lambda + \partial)[a_{\lambda} b],$$

which are called sesquilinearities.

We denote by $a_{(n)}b$ the n -th coefficient in $[a_{\lambda} b]$, i.e. $[a_{\lambda} b] = \sum_{n \in \mathbb{Z}_{\geq 0}} \lambda^n a_{(n)}b$, $a_{(n)}b \in R$. Since the formal variable λ commutes with ∂ , sesquilinearities (2.1) are equivalent to

$$(2.2) \quad [\partial a_{\lambda} b] = -\sum_{i \geq 1} \lambda^i a_{(i-1)}b \text{ and } [a_{\lambda} \partial b] = \sum_{i \geq 0} \lambda^i (\partial a_{(i)}b + a_{(i-1)}b), \text{ by letting } a_{(-1)}b := 0.$$

We also note that $\partial[a_{\lambda} b] = [\partial a_{\lambda} b] + [a_{\lambda} \partial b]$.

Definition 2.3 (Lie conformal superalgebra). A Lie conformal superalgebra R is a $\mathbb{C}[\partial]$ -module endowed with a λ -bracket $[\cdot, \cdot]_{\lambda}$ satisfying

- (i) skewsymmetry : $[b_{\lambda} a] = -p(a, b)[a_{-\partial-\lambda} b] = \sum_{j \geq 0} \frac{(-\partial-\lambda)^j}{j!} a_{(j)}b$,
(ii) Jacobi identity : $[a_{\lambda}[b_{\mu} c]] = p(a, b)[b_{\mu}[a_{\lambda} c]] + [[a_{\lambda} b]_{\lambda+\mu} c]$,

where $p(a, b) = (-1)^{p(a)p(b)}$.

Definition 2.4 (Poisson vertex algebra). A Poisson vertex algebra (PVA) is a quintuple $(\mathcal{V}, |0\rangle, \partial, \{\cdot, \cdot\}_{\lambda}, \cdot)$ satisfying the following three properties:

- (i) $(\mathcal{V}, \partial, \{\cdot, \cdot\}_{\lambda})$ is a Lie conformal superalgebra,
(ii) $(\mathcal{V}, |0\rangle, \partial, \cdot)$ is a unital differential associative commutative superalgebra,
(iii) $\{a_{\lambda} bc\} = p(a, b)b\{a_{\lambda} c\} + \{a_{\lambda} b\}c$, $a, b, c \in \mathcal{V}$.

Remark 2.5. Property (iii) in Definition 2.4 is called the left Leibniz rule. Along with the skewsymmetry of the λ -bracket, the right Leibniz rule (2.3) follows:

$$(2.3) \quad \{ab_{\lambda} c\} = \{b_{\lambda+\partial} c\}_{\rightarrow a} + \{a_{\lambda+\partial} c\}_{\rightarrow b}, \quad a, b, c \in \mathcal{V}$$

Here the small arrows in (2.3) indicate that the operator ∂ acts on the right side of the λ -brackets. For instance, $\{a_{\lambda+\partial}b\}_{\rightarrow}c = \sum_{i \geq 0} a_{(i)}b(\lambda + \partial)^i c$. As in the skewsymmetry in Definition 2.3, without arrows, the derivation ∂ acts on the left side of the λ -brackets.

Example 2.6. The Virasoro-Magri PVA on $\mathcal{V} = \mathbb{C}_{\text{diff}}[u]$ with central charge $c \in \mathbb{C}$ is defined by the λ -bracket

$$(2.4) \quad \{u_{\lambda}u\} = (\partial + 2\lambda)u + \lambda^3 c.$$

The λ -bracket on \mathcal{V} is completely determined by (2.4) using the sesquilinearities and Leibniz rules. Also, one can check the skewsymmetry and Jacobi identity by direct computations.

As in the previous example, a λ -bracket structure on a PVA is determined by λ -brackets between generating elements. If a PVA is an algebra of differential polynomials and the λ -brackets between generating elements are given, then one can find a λ -bracket between any two elements in the PVA, by the master formula (2.5).

Proposition 2.7. *Let $A = \mathbb{C}_{\text{diff}}[a_i \mid i \in I]$ be a PVA endowed with the λ -bracket $\{\cdot_{\lambda}\cdot\}$. Then A satisfies the following equation called “master formula” [4] :*

$$(2.5) \quad \{f_{\lambda}g\} = \sum_{i,j \in I, m, n \in \mathbb{Z}_{\geq 0}} \frac{\partial g}{\partial a_j^{(n)}} (\partial + \lambda)^n \{a_i_{\lambda+\partial} a_j\}_{\rightarrow} (-\partial - \lambda)^m \frac{\partial f}{\partial a_i^{(m)}}.$$

To construct an affine \mathcal{W} -algebra via a BRST complex, we recall the main ingredient called a nonlinear Lie conformal superalgebra in the remaining part of this subsection.

Let Γ_+ be a discrete additive subset of \mathbb{R}_+ containing 0 and let $\Gamma'_+ := \Gamma_+ \setminus \{0\}$. Let R be a $\mathbb{C}[\partial]$ -module endowed with the Γ_+ -grading $R = \bigoplus_{\zeta \in \Gamma'_+} R_{\zeta}$, where R_{ζ} is a $\mathbb{C}[\partial]$ -submodule. We denote by $\zeta(a)$ the Γ'_+ -grading of $a \in R$.

The tensor superalgebra $\mathcal{T}(R)$ of R is also endowed with the Γ_+ -grading

$$\mathcal{T}(R) := \bigoplus_{\zeta \in \Gamma_+} \mathcal{T}(R)[\zeta], \quad \text{where } \mathcal{T}(R)[\zeta] = \{a \in \mathcal{T}(\mathfrak{g}) \mid \zeta(a) = \zeta\},$$

such that $\zeta(a \otimes b) = \zeta(a) + \zeta(b)$ and $\zeta(1) = 0$. For $\zeta \in \Gamma'_+$, let us denote by ζ_- the largest element in Γ_+ strictly smaller than ζ and let $\mathcal{T}_{\zeta}(R)$ be the direct sum $\bigoplus_{\alpha \leq \zeta_-} \mathcal{T}(R)[\alpha]$ of $\mathbb{C}[\partial]$ -modules.

Definition 2.8. A nonlinear Lie conformal superalgebra R is a Γ'_+ -graded $\mathbb{C}[\partial]$ -module endowed with a nonlinear λ -bracket

$$[\cdot_{\lambda}\cdot] : R \otimes R \rightarrow \mathbb{C}[\lambda] \otimes \mathcal{T}(R)$$

satisfying

- (i) grading condition : $[R_{\zeta_1}{}_{\lambda}R_{\zeta_2}] \subset \mathcal{T}(R)_{\zeta_1+\zeta_2}$,
- (ii) sesquilinearity : $[\partial a_{\lambda}b] = -\lambda[a_{\lambda}b]$, $[a_{\lambda}\partial b] = (\lambda + \partial)[a_{\lambda}b]$,
- (iii) skewsymmetry : $[a_{\lambda}b] = -p(a, b)[b_{-\lambda-\partial}a]$,
- (iv) Jacobi identity : $[a_{\lambda}[b_{\mu}c]] - p(a, b)[b_{\mu}[a_{\lambda}c]] - [[a_{\lambda}b]_{\lambda+\mu}c] \in \mathbb{C}[\lambda, \mu] \otimes \mathcal{M}(R)$, where $\mathcal{M}(R)$ is the left ideal of $\mathcal{T}(R)$ generated by $a \otimes b \otimes C - p(a, b)b \otimes a \otimes C - \left(\int_{-\partial}^0 [a_{\lambda}b]d\lambda\right) \otimes C$, for $a, b, c \in R$ and $C \in \mathcal{T}(R)$.

Example 2.9. Let $p \in \mathfrak{g}$ commute with \mathfrak{n} , let $k, c \in \mathbb{C}$ and let $R = \text{Cur}_k^{\text{cp}}(\mathfrak{g})$ be the free $\mathbb{C}[\partial]$ -module $\mathbb{C}[\partial] \otimes \mathfrak{g}$ endowed with the nonlinear λ -bracket

$$[a_\lambda b] = [a, b] + \lambda k(a, b) + c(p, [a, b]), \quad \text{for } a, b \in \mathfrak{g}.$$

Take $\Gamma_+ = \mathbb{Z}_{\geq 0}$ and let $\gamma(a) = 1$ for any $a \in R$. Then R satisfies the grading condition. By computations, one can check that R is a nonlinear Lie conformal algebra. In addition, let $S(R)$ be the symmetric algebra of the vector space R over \mathbb{C} . Define the λ -bracket on $S(R)$ by the λ -bracket on R and Leibniz rules. Then one can check that $S(R)$ is a Poisson vertex algebra.

2.2. Integrable systems. Let $u_i = u_i(x, t)$, $i \in I = \{1, \dots, l\}$, be smooth functions with the coordinate $x \in S^1$ and time $t \in \mathbb{R}$ and let $u^{(n)} = (u_i^{(n)})_{i \in I} = (\frac{\partial^n}{\partial x^n} u_i)_{i \in I}$. An infinite dimensional evolution equation is a system of partial differential equations of the form

$$(2.6) \quad \frac{du_i}{dt} = P_i(u, u', u^{(2)}, \dots), \quad i \in I,$$

where P_i are in the algebra of differential polynomials

$$(2.7) \quad A = \mathbb{C}[u_i^{(n)} | i \in I, n = 0, 1, \dots] = \mathbb{C}_{\text{diff}}[u_i | i \in I].$$

Here, the derivation ∂ on A is defined by

$$(2.8) \quad \partial = \sum_{n \in \mathbb{Z}_{\geq 0}} \sum_{i=1}^l u_i^{(n+1)} \frac{\partial}{\partial u_i^{(n)}}, \quad \text{or} \quad \partial u_i^{(n)} = u_i^{(n+1)}.$$

We note that the commutator between ∂ and $\frac{\partial}{\partial u_i}$ acts on $f \in A$ as follows:

$$(2.9) \quad \left[\partial, \frac{\partial}{\partial u_i^{(n)}} \right] f = \left(\partial \frac{\partial}{\partial u_i^{(n)}} - \frac{\partial}{\partial u_i^{(n)}} \partial \right) f = - \frac{\partial f}{\partial u_i^{(n-1)}}.$$

Also, we say the total derivative order of $f \in A$ is n if the following two conditions hold:

$$(i) \text{ there is } i \in I \text{ such that } \frac{\partial f}{\partial u_i^{(n)}} \neq 0 \quad (ii) \text{ for any } j \in I, \text{ the derivative } \frac{\partial f}{\partial u_j^{(n+1)}} = 0.$$

Now we introduce integrals of motion in algebraic way and Hamiltonian systems using PVAs.

Definition 2.10 (local functional). A local functional $\int f$ is the image of $f \in A$ under the projection map

$$(2.10) \quad \int : A \rightarrow A/\partial A.$$

The space of local functionals $\int A$ is the universal space for the algebras satisfying the integration by parts.

Definition 2.11 (integral of motion). A local functional $\int f$ is called an integral of motion if

$$(2.11) \quad \int \frac{df}{dt} = 0.$$

Remark 2.12. When we need to clarify that x is the coordinate, we use ∂_x , $f(x)$, and $\int f(x)dx$ instead of ∂ , f , and $\int f$.

Equivalently, using the chain rule and the integration by parts, a functional $\int f$ is called an integral of motion if

$$(2.12) \quad \int \sum_{i \in I, n \in \mathbb{Z}_{\geq 0}} \frac{\partial f}{\partial u_i^{(n)}} \frac{du_i^{(n)}}{dt} = \int \sum_{i \in I} \left(\sum_{n \in \mathbb{Z}_{\geq 0}} (-\partial)^n \frac{\partial f}{\partial u_i^{(n)}} \right) \frac{du_i}{dt} = 0.$$

To make the formula (2.12) simpler, we recall the following definition.

Definition 2.13 (fractional derivative). The functional derivative of the differential polynomial $f \in A$ is defined as follows:

$$(2.13) \quad \frac{\delta f}{\delta u_i} = \sum_{n \in \mathbb{Z}_{\geq 0}} (-\partial)^n \frac{\partial f}{\partial u_i^{(n)}}, \text{ and } \frac{\delta f}{\delta u} = \left(\frac{\delta f}{\delta u_i} \right)_{i \in I}.$$

It is easy to see that (2.12) is equivalent to

$$(2.14) \quad \int \frac{\delta f}{\delta u} \cdot P = 0,$$

where $P = (P_i)_{i \in I}$ is same as in (2.6) and the dot product between $a = (a_i)_{i \in I}$ and $b = (b_i)_{i \in I}$ is $\sum_{i \in I} a_i b_i$. Moreover, we remark that

$$(2.15) \quad \frac{\delta}{\delta u} \circ \partial = 0.$$

In order to discuss Hamiltonian systems, let us consider an $l \times l$ matrix $H(\partial) = H_{ij}(u, u', \dots; \partial)_{i,j \in I}$, where entries are finite order differential operators in ∂ with coefficients in A . Let $\{\cdot, \cdot\}_H$ be a local Poisson brackets on A associated to $H(\partial)$, i.e.

$$\{u_i(x), u_j(y)\}_H = H_{ji}(u(y), u'(y), \dots; \partial_y) \delta(x - y).$$

Here the distribution $\delta(x - y)$ is defined as follows. (See [7], Introduction in [2] for details.)

Definition 2.14. Let $x, y \in S^1$. The distribution $\delta(x - y)$ satisfies $\int_{S^1} \phi(x) \delta(x - y) dx = \phi(y)$ for any smooth function $\phi \in A$.

Remark 2.15. One can check that $\delta(x - y) = \delta(y - x)$. Also, we let the distribution $\partial_x^n \delta(x - y)$ be defined by the equation $\int \phi(x) \partial_x^n \delta(x - y) dx = (-\partial_y)^n \phi(y)$. Then $\partial_x^n \delta(x - y) = (-\partial_y)^n \delta(x - y)$ follows by simple computations.

Using Leibniz rules of the local Poisson bracket, we have

$$(2.16) \quad \{f(x), g(y)\} = \sum_{i,j \in I, m,n \in \mathbb{Z}_{\geq 0}} \frac{\partial f(x)}{\partial u_i^{(m)}} \frac{\partial g(y)}{\partial u_j^{(n)}} \partial_x^m \partial_y^n \{u_i(x), u_j(y)\}.$$

By taking the double integral $\int \int dx dy$ of (2.16), we get

$$\left\{ \int f(x) dx, \int g(y) dy \right\} = \sum_{i,j \in I} \int \frac{\delta g(y)}{\delta u_j} H_{ji}(u(y), u'(y), \dots, \partial_y) \frac{\delta f(y)}{\delta u_i} dy$$

and, by taking $\int dx$ of both sides of (2.16), we obtain

$$\left\{ \int f(x) dx, g(y) \right\} = \sum_{i,j \in I, n \in \mathbb{Z}_{\geq 0}} \frac{\partial g(y)}{\partial u_j^{(n)}} \partial_y^n H_{ji}(u(y), u'(y), \dots, \partial_y) \frac{\delta f(y)}{\delta u_i}.$$

Definition 2.16. Let $H(\partial) = H_{ij}(u, u', \dots; \partial)_{i,j \in I}$ be an $l \times l$ matrix, where entries are finite order differential operator in ∂ with coefficients in A . The bracket $\{\cdot, \cdot\}_H : \int A \otimes \int A \rightarrow \int A$ between two local functionals is defined by

$$(2.17) \quad \left\{ \int f, \int g \right\}_H = \int \frac{\delta g}{\delta u} \left(H(\partial) \frac{\delta f}{\delta u} \right),$$

where $H(\partial) := H(u_i, u'_i, u_i^{(2)}, \dots; \partial)$, and the bracket $\{\cdot, \cdot\}_H : \int A \otimes A \rightarrow A$ between a local functional and a function is defined by

$$(2.18) \quad \left\{ \int f, g \right\}_H := \sum_{i,j \in I, n \in \mathbb{Z}_{\geq 0}} \frac{\partial g}{\partial u_j^{(n)}} \partial^n H_{ji}(\partial) \frac{\delta f}{\delta u_i},$$

where $H(\partial) := (H_{ij}(\partial))_{i,j \in I}$.

Note that the brackets in Definition 2.16 are well-defined by (2.15). Also, If we let $H_{ji}(\lambda) = \{u_i \lambda u_j\}_H$, by (2.17), (2.18) and (2.5), we have

$$(2.19) \quad \left\{ \int f, \int g \right\}_H = \left\{ \int f \lambda g \right\}_H |_{\lambda=0}, \quad \left\{ \int f, \int g \right\}_H = \int \left\{ \int f, g \right\}_H.$$

Definition 2.17. (i) An $l \times l$ matrix-valued differential operator $H(\partial)$ is called a Poisson structure if

$$(2.20) \quad H_{ji}(\lambda) := \{u_i \lambda u_j\}_H$$

gives a PVA structure on A .

(ii) Let $H(\partial)$ be a Poisson structure. An evolution equations of the form

$$(2.21) \quad \frac{du}{dt} = H(\partial) \frac{\delta h}{\delta u}$$

is called a Hamiltonian system. Equivalently, in terms of a λ -bracket on A

$$(2.22) \quad \frac{du}{dt} = \{h \lambda u\}_H |_{\lambda=0}$$

is called a Hamiltonian system.

(iii) Local functionals $\int h, \int h' \in A$ are called to be in involution if $\{\int h, \int h'\}_H = 0$ and the Hamiltonian system (2.22) is called an integrable system if there are infinitely many linearly independent local functionals $\int h_0, \dots, \int h_n, n = 1, 2, \dots$, which are in involution.

One can see that if $\int h_n$ for $n \in \mathbb{Z}_{\geq 0}$ are linearly independent and they are all in involution, then the hierarchy of Hamiltonian equations

$$\left(\frac{du}{dt} = \{h_n \lambda u\}_H |_{\lambda=0} \right)_{n \in \mathbb{Z}_{\geq 0}},$$

consists of integrable equations. Hence the following observation called Lenard scheme plays a key role in finding integrals of motion.

Lenard Scheme. Let $H(\partial)$ and $K(\partial)$ be Poisson structures such that

$$H_{ji}(\lambda) := \{u_i \lambda u_j\}_H, \quad K_{ji}(\lambda) := \{u_i \lambda u_j\}_K.$$

If there are local functions $\int h_0, \int h_1, \dots, \int h_n$ satisfying

$$(2.23) \quad \begin{aligned} \{h_0 \lambda u_i\}_K |_{\lambda=0} &= 0, \\ \{h_t \lambda u_i\}_H |_{\lambda=0} &= \{h_{t+1} \lambda u_i\}_K |_{\lambda=0}, \quad t = 0, \dots, n-1, \end{aligned}$$

for $i \in I$, then $\int h_0, \dots, \int h_n$ are all in involution with respect to both brackets.

Theorem 2.18. [2] Let $H(\partial)$ and $K(\partial)$ be a bi-Hamiltonian pair, i.e. $(H + K)(\partial)$ is also a Hamiltonian operator on A^l . If the following conditions hold:

- (i) $K(\partial)$ is non-degenerate,
- (ii) $H(\partial) \frac{\delta h_0}{\delta u} = K(\partial) \frac{\delta h_1}{\delta u}$,
- (iii) $\left(\text{span}_{\mathbb{C}} \left\{ \frac{\delta h_0}{\delta u}, \frac{\delta h_1}{\delta u} \right\} \right)^{\perp} \subset K(\partial)$,

then there exists a sequence of local functionals $\int h_i$, $i \in \mathbb{Z}_{\geq 0}$, satisfying (2.23).

Example 2.19 (KdV hierarchy). (See [2] for details)

Let $R = \mathbb{C}_{\text{diff}}[u]$ be a PVA with two λ -brackets $\{\cdot, \cdot\}_H$ and $\{\cdot, \cdot\}_K$ such that

$$\{u_\lambda u\}_H = (\partial + 2\lambda)u + c\lambda^3, \quad \{u_\lambda u\}_K = \lambda, \quad c \in \mathbb{C}$$

which means that $H(\partial) = u' + 2u\partial + c\partial^3$ and $K(\partial) = \partial$. Then we have

$$(2.24) \quad \begin{aligned} \{u_\lambda u\}_K |_{\lambda=0} &= 0, \quad \{u_\lambda u\}_H |_{\lambda=0} = \left\{ \frac{1}{2} u_\lambda^2 u \right\}_K |_{\lambda=0} = \partial u, \\ \left\{ \frac{1}{2} u_\lambda^2 u \right\}_H |_{\lambda=0} &= \left\{ \frac{1}{2} u^3 + \frac{1}{2} cuu'' u \right\}_K |_{\lambda=0} = 3uu' + cu'''. \end{aligned}$$

Hence $h_0 = u$, $h_1 = \frac{1}{2}u^2$, $h_2 = \frac{1}{2}u^3 + \frac{1}{2}cuu''$ satisfy (2.23). Recursively, one can find h_i for any $i \in \mathbb{Z}_{\geq 0}$ and the equations $\frac{du}{dt} = \{h_i \lambda u\}_K |_{\lambda=0}$ are all integrable systems. Especially, $\frac{du}{dt} = \{h_2 \lambda u\}_K |_{\lambda=0} = 3uu' + cu'''$ is the KdV equation.

3. TWO EQUIVALENT DEFINITIONS OF CLASSICAL AFFINE \mathcal{W} -ALGEBRAS

Let \mathfrak{g} be a finite dimensional simple Lie algebra over \mathbb{C} with an \mathfrak{sl}_2 triple (e, h, f) . The Lie algebra \mathfrak{g} can be decomposed into direct sum of eigenspaces of $\text{ad} \frac{h}{2}$, $\mathfrak{g} = \bigoplus_{i \in \frac{\mathbb{Z}}{2}} \mathfrak{g}(i)$, where $\mathfrak{g}(i) = \{g \in \mathfrak{g} | [\frac{h}{2}, g] = ig\}$. Let

$$\mathfrak{n} = \bigoplus_{i \geq \frac{1}{2}} \mathfrak{g}(i) \subset \mathfrak{m} = \bigoplus_{i \geq 1} \mathfrak{g}(i)$$

be two nilpotent subalgebras of \mathfrak{g} and let

$$\mathfrak{g}_f = \{g \in \mathfrak{g} \mid [g, f] = 0\}.$$

We fix a symmetric bilinear form (\cdot, \cdot) such that $(e, f) = 1$ and $(h, h) = 2$.

3.1. The first definition of classical affine \mathcal{W} -algebras. The following three nonlinear Lie conformal superalgebras are needed to define classical affine \mathcal{W} -algebras.

- (1) Let $p \in \mathfrak{g}$ commute with \mathfrak{n} , let $k, c \in \mathbb{C}$ and let $Cur_k^{cp}(\mathfrak{g})$ be the free $\mathbb{C}[\partial]$ -module $\mathbb{C}[\partial] \otimes \mathfrak{g}$ endowed with the nonlinear λ -bracket

$$[a_\lambda b] = [a, b] + \lambda k(a, b) + c(p, [a, b]), \text{ for } a, b \in \mathfrak{g}.$$

- (2) Let Π be the parity reversing map and let $\phi_{\mathfrak{n}}$ and $\phi_{\mathfrak{n}^*}$ be purely odd vector superspaces isomorphic to $\Pi \mathfrak{n}$ and $\Pi \mathfrak{n}^*$, respectively. The charged free fermion nonlinear Lie conformal superalgebra F_{ch} is the free $\mathbb{C}[\partial]$ -module $\mathbb{C}[\partial] \otimes (\phi_{\mathfrak{n}} \oplus \phi_{\mathfrak{n}^*})$ endowed with the nonlinear λ -bracket defined by

$$\begin{aligned} [\phi_a \lambda \phi^\theta] &= \theta(a), \quad a \in \mathfrak{n}, \quad \theta \in \mathfrak{n}^*, \\ [\phi_a \lambda \phi_b] &= [\phi^{\theta_1} \lambda \phi^{\theta_2}] = 0, \quad a, b \in \mathfrak{n}, \quad \theta_1, \theta_2 \in \mathfrak{n}^*. \end{aligned}$$

- (3) Let $\Phi_{\mathfrak{n}/\mathfrak{m}}$ be a vector space isomorphic to $\mathfrak{g}(\frac{1}{2})$ with the skew symmetric bilinear form $\langle \Phi_{[a]}, \Phi_{[b]} \rangle = (f, [a, b])$. The Neutral free fermion nonlinear Lie conformal algebra F_{ne} is the free $\mathbb{C}[\partial]$ -module $\mathbb{C}[\partial] \otimes \Phi_{\mathfrak{n}/\mathfrak{m}}$ endowed with the nonlinear λ -bracket

$$[\Phi_{[a]} \lambda \Phi_{[b]}] = \langle \Phi_{[a]}, \Phi_{[b]} \rangle = (f, [a, b]), \quad [a], [b] \in \mathfrak{n}/\mathfrak{m}.$$

We denote the direct sum of the three nonlinear Lie conformal superalgebras $Cur_k^{cp}(\mathfrak{g})$, F_{ch} and F_{ne} by

$$(3.1) \quad R_k^{cp}(\mathfrak{g}, f) := \mathbb{C}[\partial] \otimes (\mathfrak{g} \oplus \phi_{\mathfrak{n}} \oplus \phi_{\mathfrak{n}^*} \oplus \Phi_{\mathfrak{n}/\mathfrak{m}}).$$

Let $S(R_k^{cp}(\mathfrak{g}, f))$ be the Poisson vertex algebra endowed with the λ -bracket such that

$$\{a_\lambda b\} = [a_\lambda b], \quad a, b \in R_k^{cp}(\mathfrak{g}, f),$$

and extended by left and right Leibniz rules.

Let $X : \mathfrak{n} \rightarrow R_k^{cp}(\mathfrak{g}, f)$ be a linear map such that $X(a) = a + (f, a) + \Phi_{[a]}$, $a \in \mathfrak{n}$, and take the odd element

$$\begin{aligned} (3.2) \quad d &= \sum_{\alpha \in S} \phi^{v^\alpha} (u_\alpha + (f, u_\alpha) + \Phi_{[u_\alpha]}) + \frac{1}{2} \sum_{\alpha, \beta \in S} \phi^{v^\alpha} \phi^{v^\beta} \phi_{[u_\beta, u_\alpha]} \\ &= \sum_{\alpha \in S} \phi^{v^\alpha} X_{u_\alpha} + \frac{1}{2} \sum_{\alpha, \beta \in S} \phi^{v^\alpha} \phi^{v^\beta} \phi_{[u_\beta, u_\alpha]} \end{aligned}$$

in $S(R_k^{cp}(\mathfrak{g}, f))$, where $\{u_\alpha \mid \alpha \in S\}$ and $\{v^\alpha \mid \alpha \in S\}$ are dual bases of \mathfrak{n} and \mathfrak{n}^* . We also note that

$$\{X_a \lambda X_b\} = X_{[a, b]}.$$

Proposition 3.1. *Let the operator $d_{(0)}$ on $S(R_k^{cp}(\mathfrak{g}, f))$ be defined by $d_{(0)}(a) = \{d_\lambda a\}|_{\lambda=0}$. Then $d_{(0)}$ is a differential on the complex $S(R_k^{cp}(\mathfrak{g}, f))$.*

Proof. Since d is an odd element in $S(R_k^{cp}(\mathfrak{g}, f))$, we have $\{d_\lambda\{d_\mu A\}\} = -\{d_\lambda\{d_\mu A\}\} + \{\{d_\lambda d\}_{\lambda+\mu} A\}$. This implies that if $\{d_\lambda d\} = 0$, then $d_{(0)}^2 A = \frac{1}{2}\{\{d_\lambda d\}_{\lambda+\mu} A\}|_{\lambda=\mu=0} = 0$. Hence it suffices to show that $\{d_\lambda d\} = 0$.

Let us compute $\{d_\lambda d\}$ using Jacobi identity and the skewsymmetry. The λ -brackets between the summands of d are as follows:

(i)

$$\sum_{\alpha, \beta \in S} \{\phi^{v^\alpha} X_{u_\alpha} \lambda \phi^{v^\beta} X_{u_\beta}\} = \sum_{\alpha, \beta \in S} \phi^{v^\beta} \phi^{v^\alpha} X_{[u_\beta, u_\alpha]},$$

(ii)

$$\begin{aligned} & \sum_{\alpha, \beta, \gamma \in S} \{\phi^{v^\alpha} X_{u_\alpha} \lambda \phi^{v^\beta} \phi^{v^\gamma} \phi_{[u_\gamma, u_\beta]}\} \\ &= \sum_{\alpha, \beta, \gamma \in S} \phi^{v^\beta} \phi^{v^\gamma} v^\alpha([u_\gamma, u_\beta]) X_{u_\alpha} = \sum_{\alpha, \beta, \gamma \in S} \phi^{v^\beta} \phi^{v^\gamma} X_{[u_\gamma, u_\beta]}, \end{aligned}$$

(iii)

$$\begin{aligned} & \sum_{\alpha, \beta, \gamma, \delta \in S} \{\phi^{v^\alpha} \phi^{v^\beta} \phi_{[u_\beta, u_\alpha]} \lambda \phi^{v^\gamma} \phi^{v^\delta} \phi_{[u_\delta, u_\gamma]}\} \\ &= 4 \sum_{\alpha, \beta, \gamma, \delta \in S} v^\delta([u_\beta, u_\alpha]) \phi^{v^\gamma} \phi^{v^\alpha} \phi^{v^\beta} \phi_{[u_\gamma, u_\delta]} = 4 \sum_{\alpha, \beta, \gamma \in S} \phi^{v^\gamma} \phi^{v^\alpha} \phi^{v^\beta} \phi_{[u_\gamma, [u_\beta, u_\alpha]]} = 0. \end{aligned}$$

The last equality in (iii) follows from:

$$\phi^{v^\gamma} \phi^{v^\alpha} \phi^{v^\beta} \phi_{[u_\gamma, [u_\beta, u_\alpha]]} + \phi^{v^\alpha} \phi^{v^\beta} \phi^{v^\gamma} \phi_{[u_\alpha, [u_\gamma, u_\beta]]} + \phi^{v^\beta} \phi^{v^\gamma} \phi^{v^\alpha} \phi_{[u_\beta, [u_\alpha, u_\gamma]]} = 0.$$

Hence we have

$$\{d_\lambda d\} = (i) + (ii) + \frac{1}{4}(iii) = \sum_{\alpha, \beta, \gamma \in S} \phi^{v^\gamma} \phi^{v^\alpha} \phi^{v^\beta} \phi_{[u_\gamma, [u_\beta, u_\alpha]]} = 0$$

and $d_{(0)}$ is a differential on the complex $S(R_k^{cp}(\mathfrak{g}, f))$. □

Definition 3.2 (classical affine \mathcal{W} -algebra (1)). Let $C_1 = S(R_k^{cp}(\mathfrak{g}, f))$ and $d_1 = d_{(0)}$, where d is defined in (3.2). The cohomology $H(C_1, d_1)$ is the classical affine \mathcal{W} -algebra $\mathcal{W}_1(\mathfrak{g}, k, f)$ associated to \mathfrak{g} , f and $k \in \mathbb{C}$.

Remark 3.3. (i) Consider the PVA isomorphism

$$(3.3) \quad \psi^{cp} : S(R_k^0(\mathfrak{g}, f)) \rightarrow S(R_k^{cp}(\mathfrak{g}, f))$$

such that $\psi^{cp}(a) = \begin{cases} a + [cp, a] & \text{if } a \in \mathfrak{g}, \\ a & \text{if } a \in \phi_n \oplus \phi^{n*} \oplus \Phi_{\mathfrak{n}/\mathfrak{m}}. \end{cases}$ Then ψ^{cp} induces the well-defined PVA isomorphism $\overline{\psi^{cp}} : H(S(R_k^0(\mathfrak{g}, f)), d_1) \rightarrow H(S(R_k^{cp}(\mathfrak{g}, f)), d_1)$. Hence the PVA structure on $H(S(R_k^{cp}(\mathfrak{g}, f)), d_1)$ is independent on c and p .

- (ii) Let $\{u_\alpha | \alpha \in \bar{S}\}$ and $\{u^\alpha | \alpha \in \bar{S}\}$ be dual bases of \mathfrak{g} , where $u_\alpha \in \mathfrak{g}(j_\alpha)$. Let $\{z_1, \dots, z_{2s}\}$ and $\{z_1^*, \dots, z_{2s}^*\}$ be bases of $\mathfrak{g}(\frac{1}{2})$ such that $(f, [z_i, z_j^*]) = \delta_{i,j}$. Then there is an energy-momentum field \bar{L} of $\mathcal{W}_1(\mathfrak{g}, k, f)$ which is the projection of the element

$$(3.4) \quad L = \frac{1}{2k} \sum_{\alpha \in \bar{S}} u_\alpha u^\alpha + \partial x - \sum_{\alpha \in S} j_\alpha \phi^{v_\alpha} \partial \phi_{u_\alpha} + \sum_{\alpha \in S} (1 - j_\alpha) \partial \phi^{v_\alpha} \phi_{u_\alpha} + \frac{1}{2} \sum_{i=1}^{2s} \partial \Phi_{[z_i^*]} \Phi_{[z_i]}$$

of $S(R_k^{cp}(\mathfrak{g}, f))$. By direct computations, we have $\{\bar{L}_\lambda \bar{L}\} = (\partial + 2\lambda) \bar{L} - \frac{1}{2} k \lambda^3$.

3.2. The second definition of classical affine \mathcal{W} -algebras. Let $S(\mathbb{C}[\partial] \otimes \mathfrak{g})$ be the symmetric algebra generated by $\mathbb{C}[\partial] \otimes \mathfrak{g}$ endowed with the λ -bracket induced from the λ -bracket on $Cur_k^{cp}(\mathfrak{g})$ and Leibniz rules. Consider the differential algebra ideal

$$(3.5) \quad J = \langle m - \chi(m) | m \in \mathfrak{m}, \chi(m) = (f, m) \rangle$$

of $S(\mathbb{C}[\partial] \otimes \mathfrak{g})$. The λ -adjoint action of \mathfrak{n} on $S(\mathbb{C}[\partial] \otimes \mathfrak{g})/J$ is defined by

$$(\text{ad}_\lambda n)(a + J) = \{n_\lambda a\} + J[\lambda].$$

For given $n \in \mathfrak{n}$, $x \in \mathfrak{m}$ and $A \in S(\mathbb{C}[\partial] \otimes \mathfrak{g})$, we have

$$\{n_\lambda A(m - \chi(m))\} = A\{n_\lambda m - \chi(m)\} + \{n_\lambda A\}(m - \chi(m)) \in J[\lambda].$$

Hence the $\text{ad}_\lambda \mathfrak{n}$ action is well-defined and we can define the vector space:

$$\mathcal{W}_2(\mathfrak{g}, k, f) = (S(\mathbb{C}[\partial] \otimes \mathfrak{g}) / \langle m + \chi(m) : m \in \mathfrak{m} \rangle)^{\text{ad}_\lambda \mathfrak{n}}.$$

The following proposition shows that the multiplication and the Poisson λ -bracket on $\mathcal{W}_2(\mathfrak{g}, k, f)$ can be defined by $(a + J) \cdot (b + J) = ab + J$ and $\{a + J, b + J\} = \{a_\lambda b\} + J[\lambda]$.

Proposition 3.4. *The multiplication and the Poisson λ -bracket are well-defined on $\mathcal{W}_2(\mathfrak{g}, k, f)$.*

Proof. Let $a + J, b + J \in \mathcal{W}_2(\mathfrak{g}, k, f)$ and $n \in \mathfrak{n}$. Then we have

$$(\text{ad}_\lambda n)((a + J) \cdot (b + J)) = (b + J)((\text{ad}_\lambda n)(a + J)) + (a + J)((\text{ad}_\lambda n)(b + J)) \in 0 + J[\lambda]$$

by the definition of $\mathcal{W}_2(\mathfrak{g}, k, f)$. Hence the multiplication is well-defined on $\mathcal{W}_2(\mathfrak{g}, k, f)$.

Moreover, one can check that $\{a + A(m_1 - \chi(m_1))_\lambda b + B(m_2 - \chi(m_2))\} \in \{a_\lambda b\} + J[\lambda]$, for any $A, B \in S(\mathbb{C}[\partial] \otimes \mathfrak{g})$, and $(\text{ad}_\lambda n)\{a_\mu b\} + J[\mu] \subset 0 + J[\lambda, \mu]$, by Jacobi identity. Hence the Poisson λ -bracket is also well-defined on $\mathcal{W}_2(\mathfrak{g}, k, f)$. \square

Definition 3.5 (classical affine \mathcal{W} -algebra (2)). The classical affine \mathcal{W} -algebra associated to \mathfrak{g} , f and k is the PVA

$$\mathcal{W}_2(\mathfrak{g}, f, k) = (S(\mathbb{C}[\partial] \otimes \mathfrak{g}) / \langle m + \chi(m) | m \in \mathfrak{m} \rangle)^{\text{ad}_\lambda \mathfrak{n}}.$$

Remark 3.6. Let $(\mathcal{W}^{(cp)}(\mathfrak{g}, f, k), \{\cdot_\lambda \cdot\}^{(cp)})$, where $c \in \mathbb{C}$ and $p \in \mathfrak{z}(\mathfrak{n})$, be the classical affine \mathcal{W} -algebra associated to \mathfrak{g} and f endowed with the Poisson bracket induced from the bracket $\{\cdot_\lambda \cdot\}^{(cp)}$ on $S(\mathbb{C}[\partial] \otimes \mathfrak{g})$ such that

$$\{a_\lambda b\}^{(cp)} = [a, b] + c(p, [a, b]) + \lambda k(a, b), \quad a, b \in \mathfrak{g}.$$

We have the PVA isomorphism

$$(3.6) \quad \psi^{(cp)} : (S(\mathbb{C}[\partial] \otimes \mathfrak{g}), \{\cdot_\lambda \cdot\}^{(0)}) \simeq (S(\mathbb{C}[\partial] \otimes \mathfrak{g}), \{\cdot_\lambda \cdot\}^{(cp)})$$

defined by $\psi^{(cp)}(a) = a + [cp, a]$, $a \in \mathfrak{g}$. Moreover, $\psi^{(cp)}$ induces the well-defined PVA isomorphism between $\mathcal{W}^{(0)}(\mathfrak{g}, f, k)$ and $\mathcal{W}^{(cp)}(\mathfrak{g}, f, k)$. Hence the Poisson algebra structure on $\mathcal{W}(\mathfrak{g}, k, f)$ is independent on c and p .

3.3. Equivalence of the two definitions of classical affine \mathcal{W} -algebras. In this section, we assume $p = 0$. Given $a \in \mathfrak{g}$, let us denote

$$(3.7) \quad J_a := a + \sum_{\alpha \in S} \phi^{v^\alpha} \phi_{\{u_\alpha, a\}}.$$

Then

$$S(\mathbb{C}[\partial] \otimes (\mathfrak{g} \oplus \phi_{\mathfrak{n}} \oplus \phi^{\mathfrak{n}^*} \oplus \Phi_{\mathfrak{n}/\mathfrak{m}})) = S(\mathbb{C}[\partial] \otimes (J_{\mathfrak{g}} \oplus \phi_{\mathfrak{n}} \oplus \phi^{\mathfrak{n}^*} \oplus \Phi_{\mathfrak{n}/\mathfrak{m}})).$$

The Poisson λ -brackets between J_a and elements in $R_k^0 = J_{\mathfrak{g}} \oplus \phi_{\mathfrak{n}} \oplus \phi^{\mathfrak{n}^*} \oplus \Phi_{\mathfrak{n}/\mathfrak{m}}$ are as follows:

$$(3.8) \quad \begin{aligned} \{J_a \lambda \phi_n\} &= \{a + \sum_{\alpha \in S} \phi^{v^\alpha} \phi_{[u_\alpha, a]} \lambda \phi_n\} = - \sum_{\alpha \in S} v^\alpha(n) \phi_{[u_\alpha, a]} = \phi_{[a, n]}, \\ \{J_a \lambda \phi^\theta\} &= \{a + \sum_{\alpha \in S} \phi^{v^\alpha} \phi_{[u_\alpha, a]} \lambda \phi^\theta\} = \sum_{\alpha \in S} \phi^{v^\alpha} \{\phi_{[u_\alpha, a]} \lambda \phi^\theta\} = \phi^{a \cdot \theta}, \\ \{J_a \lambda \Phi_{[n]}\} &= 0, \\ \{J_a \lambda J_b\} &= J_{[a, b]} + k\lambda(a, b) + \sum_{\alpha \in S} \phi^{v^\alpha} (-\phi_{[\pi_{\leq}[u_\alpha, a], b]} + \phi_{[\pi_{\leq}[u_\alpha, b], a]}), \end{aligned}$$

where $\pi_{\leq} : \mathfrak{g} \rightarrow \bigoplus_{i \leq 0} \mathfrak{g}(i)$ is the projection map and $a, b \in \mathfrak{g}$, $n \in \mathfrak{n}$, $\theta \in \mathfrak{n}^*$. Equation (3.8) shows that if a and b are both in $\bigoplus_{i \geq 0} \mathfrak{g}(i)$ or both in $\bigoplus_{i \leq 0} \mathfrak{g}(i)$ then

$$(3.9) \quad \{J_a \lambda J_b\} = J_{[a, b]} + \lambda k(a, b).$$

Let $J_{\leq} = \{J_a | a \in \bigoplus_{i \leq 0} \mathfrak{g}(i)\}$ and let $r_+ = \phi_{\mathfrak{n}} \oplus d_{(0)} \phi_{\mathfrak{n}}$, $r_- = J_{\leq} \oplus \phi^{\mathfrak{n}^*} \oplus \Phi_{\mathfrak{n}/\mathfrak{m}}$, $R_+ = \mathbb{C}[\partial] \otimes r_+$, and $R_- = \mathbb{C}[\partial] \otimes r_-$. Since

$$(3.10) \quad \{d \lambda J_a\} = \sum_{\alpha \in S} \left(\phi^{v^\alpha} (J_{\pi_{\leq}[u_\alpha, a]} - \Phi_{[u_\alpha, a]} - (u_\alpha, [a, f])) + k(\partial + \lambda)(u_\alpha, a) \phi^{v^\alpha} \right)$$

and $d_{(0)}(\phi_a) = J_a + (a, f) + \Phi_{[a]}$, we have

$$d_{(0)}(S(R_+)) \subset S(R_+) \text{ and } d_{(0)}(S(R_-)) \subset S(R_-).$$

Hence we can define cohomologies $H(S(R_+), d_+)$ and $H(S(R_-), d_-)$, where $d_+ = d_{(0)}|_{S(R_+)}$ and $d_- = d_{(0)}|_{S(R_-)}$. By K nneth lemma, we have $H(C_1, d_{(0)}) = H(S(R_+), d_+) \otimes H(S(R_-), d_-)$. Moreover, it is easy to see that $\ker(d_+) = \text{im}(d_+) = d_{(0)} \phi_{\mathfrak{n}}$ which implies that $H(S(R_+), d_+) = S(H(R_+, d_+)) = \mathbb{C}$ and

$$(3.11) \quad H(C_1, d_1) = H(S(R_-), d_-) = S(H(R_-, d_-)).$$

Let us define the degree and the charge on R_- as follows:
(3.12)

$$\begin{aligned} \deg(J_a) &= -j + \frac{1}{2}, \quad \deg(\phi^{v^\alpha}) = j_\alpha - \frac{1}{2}, \quad a \in \mathfrak{g}(j), \quad u_\alpha \in \mathfrak{g}(j_\alpha), \\ \deg(\Phi_{[n]}) &= 0, \quad \deg(\partial) = 0, \quad n \in \mathfrak{n}, \end{aligned}$$

$$\text{charge}(J_a) = \text{charge}(\Phi_{[n]}) = \text{charge}(\partial) = 0, \text{charge}(\phi^\theta) = 1, \quad a \in \bigoplus_{i \leq 0} \mathfrak{g}(i), n \in \mathfrak{n}, \theta \in \mathfrak{n}^*.$$

We denote by $\{F_i\}_{i \in \frac{\mathbb{Z}}{2}}$ the increasing filtration with respect to the degree on $S(R_-)$:

$$(3.13) \quad \cdots \subset F_{p-\frac{1}{2}}(S(R_-)) \subset F_p(S(R_-)) \subset F_{p+\frac{1}{2}}(S(R_-)) \subset \cdots,$$

where $F_p(S(R_-)) = \bigoplus_{i \leq p} S(R_-)(i)$ and $S(R_-)(i) = \{A \in S(R_-) | \deg(A) = i\}$.

Then we have the graded superalgebra $\text{gr}S(R_-) := \bigoplus_{i \in \frac{\mathbb{Z}}{2}} F_{i+\frac{1}{2}}(S(R_-))/F_i(S(R_-))$ of $S(R_-)$ and the graded differential $\text{gr}d_- : \text{gr}S(R_-) \rightarrow \text{gr}S(R_-)$ which is induced from the differential d_- on $S(R_-)$. Since $\text{im}(\text{gr}d_-|_{S(R_-)}) = \mathbb{C}[\partial] \otimes \phi^{\mathfrak{n}^*}$ and $\ker(\text{gr}d_-|_{S(R_-)}) = \mathbb{C}[\partial] \otimes \phi^{\mathfrak{n}^*} \oplus \mathbb{C}[\partial] \otimes \langle J_a | a \in \mathfrak{g}_f \rangle$, by Künneth lemma, we obtain

$$(3.14) \quad H(\text{gr}S(R_-), \text{gr}d_-) = H^0(\text{gr}S(R_-), \text{gr}d_-) = S(\mathbb{C}[\partial] \otimes \langle J_a | a \in \mathfrak{g}_f \rangle).$$

Proposition 3.7. *Let $\{u_i | i \in I\}$ be a basis of \mathfrak{g}_f consisting of eigenvectors of $\text{ad}_{\frac{h}{2}}$ and assume that $u_i \in \mathfrak{g}(\frac{n_i}{2})$. Then there exists $A_{u_i} \in F_{-\frac{n_i}{2}}(S(R_-))$ with charge 0 such that $H(S(R_-), d_-)$ is generated by*

$$\{J_{u_i} + A_{u_i}\}_{i \in I}$$

as a differential algebra.

Proof. Let L be the energy-momentum field introduced in (3.4). The complex $S(R_-)$ is locally finite, since, for each $i \in \frac{\mathbb{Z}}{2}$, the subspace

$$L_i := \{a \in S(R_-) | \{L_\lambda a\} = (\partial + i\lambda)a + o(\lambda)\}$$

is finite dimensional and $d_-|_{L_i} \subset L_i$. Hence we have

$$H(\text{gr}S(R_-), \text{gr}d_-) \simeq \text{gr}H(S(R_-), d_-).$$

Recall that $H(\text{gr}S(R_-), \text{gr}d_-) = S(\mathbb{C}[\partial] \otimes \langle J_a | a \in \mathfrak{g}_f \rangle)$ and $\deg(J_{u_i}) = -\frac{n_i}{2} + \frac{1}{2}$. Thus the cohomology $H(S(R_-), d_-)$ is generated by the set of the form $\{J_{u_i} + A_{u_i}\}_{i \in I}$, where $A_{u_i} \in F_{-\frac{n_i}{2}}(S(R_-))$. Moreover, by equation (3.14), any element in $H(S(R_-), d_-)$ has a representative of charge 0. Hence we can choose A_{u_i} with charge 0. □

The next theorem is our main purpose of this section.

Theorem 3.8. *Let $i : S(R_-) \rightarrow S(\mathbb{C}[\partial] \otimes \mathfrak{g}) / \langle m - \chi(m) | m \in \mathfrak{m} \rangle$ be an associative superalgebra homomorphism defined by*

$$(3.15) \quad \partial^k J_a \mapsto \partial^k a, \quad \partial^k \Phi_b \mapsto -\partial^k b, \quad \partial^k \phi^\theta \mapsto 0$$

where $k \in \mathbb{Z}_{\geq 0}$, $a \in \bigoplus_{i \leq 0} \mathfrak{g}(i)$, $b \in \mathfrak{g}(\frac{1}{2})$ and $\theta \in \mathfrak{n}^$. Then*

(1) the homomorphism i induces an associative algebra isomorphism

$$(3.16) \quad j : H(S(R_-), d_-) \rightarrow (S(\mathbb{C}[\partial] \otimes \mathfrak{g}) / \langle m - \chi(m) : m \in \mathfrak{m} \rangle)^{\text{ad}_{\lambda^n}}.$$

(2) the isomorphism (3.16) is a Poisson vertex algebra isomorphism.

Hence $\mathcal{W}_1(\mathfrak{g}, f, k)$ and $\mathcal{W}_2(\mathfrak{g}, f, k)$ are isomorphic as PVAs and we denote the classical affine \mathcal{W} -algebra associated to \mathfrak{g} , f and k by $\mathcal{W}(\mathfrak{g}, f, k)$.

Proof. In this proof, we use the following notations.

Notation 3.9. We denote

$$\begin{aligned} A_i &:= a_{i,1} \cdots a_{i,k_i}, N_i := n_{i,1} \cdots n_{i,l_i}, G := g_1 g_2 \cdots g_m, \\ \partial^{R_i} A_i &:= \partial^{r_{i,1}} a_{i,1} \cdots \partial^{r_{i,k_i}} a_{i,k_i}, \partial^{S_i} N_i := \partial^{s_{i,1}} n_{i,1} \cdots \partial^{s_{i,l_i}} n_{i,l_i}, \partial^T G := \partial^{t_1} g_1 \cdots \partial^{t_m} g_m \\ \text{for } a_{i,j} &\in \bigoplus_{i \leq 0} \mathfrak{g}(i), n_{i,j} \in \mathfrak{g}(\tfrac{1}{2}), g_i \in \bigoplus_{\alpha \leq 1} \mathfrak{g}_\alpha \text{ and } R_i = (r_{i,1}, \dots, r_{i,k_i}), S_i = (s_{i,1}, \dots, s_{i,l_i}), T = (t_1, \dots, t_m). \text{ Also, let} \\ J_{A_i} &:= J_{a_{i,1}} \cdots J_{a_{i,k_i}}, \partial^{R_i} J_{A_i} := \partial^{r_{i,1}} J_{a_{i,1}} \cdots \partial^{r_{i,k_i}} J_{a_{i,k_i}}, \\ \Phi_{N_i} &:= \Phi_{[n_{i,1}]} \cdots \Phi_{[n_{i,l_i}]}, \partial^{S_i} \Phi_{N_i} := \partial^{s_{i,1}} \Phi_{[n_{i,1}]} \cdots \partial^{s_{i,l_i}} \Phi_{[n_{i,l_i}]}, \\ K_{g_i} &:= J_{g_i} - \Phi_{[g_i]} - (g_i, f), K_G := K_{g_1} \cdots K_{g_m}, \partial^T K_G := \partial^{t_1} K_{g_1} \cdots \partial^{t_m} K_{g_m}. \end{aligned}$$

Since $i(\phi^{n^*}) = 0$, we have $i(d_-(S(R_-))) = 0$. Hence the homomorphism i induces the well-defined associative algebra homomorphism

$$\bar{i} : H(S(R_-), d_-) \rightarrow S(\mathbb{C}[\partial] \otimes \mathfrak{g}) / \langle m - \chi(m) | m \in \mathfrak{m} \rangle.$$

By Proposition 3.7, any element in $H(S_-, d_-)$ has a unique representative of the form $\sum_{i \in I} \partial^{R_i} J_{A_i} \partial^{S_i} \Phi_{N_i}$. Hence \bar{i} is injective.

Let us show that the image under \bar{i} is $(S(\mathbb{C}[\partial] \otimes \mathfrak{g}) / \langle m - \chi(m) | m \in \mathfrak{m} \rangle)^{\text{ad}_{\lambda^n}}$, i.e.

$$(3.17) \quad d_- \left(\sum_{i \in I} \partial^{R_i} J_{A_i} \partial^{S_i} \Phi_{N_i} \right) = 0 \text{ if and only if } \{n_\lambda \sum_{i \in I} \partial^{R_i} A_i \partial^{S_i} N_i\} = 0.$$

We notice that $K_g = J_g$ if $g \in \bigoplus_{i \leq 0} \mathfrak{g}(i)$ and $K_g = -\Phi_{[g]}$ if $g \in \mathfrak{g}(\tfrac{1}{2})$. Hence any element in $H(C_1, d_1)$ can be written as a polynomial in $\partial^t K_g$ where $g \in \bigoplus_{i \leq 1} \mathfrak{g}(i)$ and $t \in \mathbb{Z}_{\geq 0}$. By (3.10), we obtain

$$(3.18) \quad d_-(K_g) = \sum_{\alpha \in S} \phi^{v_\alpha} K_{\{u_\alpha, g\}} + k \sum_{\alpha \in S} \partial \phi^{v_\alpha}(u_\alpha, a).$$

Since d is a derivation, by equation (3.18), we have

$$(3.19) \quad \begin{aligned} d_-(\partial^T K_G) &= \sum_{j=1}^m \partial^{t_j} \left(\sum_{\alpha \in S} \phi^{v_\alpha} K_{\{u_\alpha, g_j\}} + k \sum_{\alpha \in S} \partial \phi^{v_\alpha}(u_\alpha, g_j) \right) \\ &\quad \cdot \partial^{t_1} K_{g_1} \cdots \partial^{t_{j-1}} K_{g_{j-1}} \partial^{t_{j+1}} K_{g_{j+1}} \cdots \partial^{t_m} K_{g_m}. \end{aligned}$$

Here,

$$\begin{aligned}
& \partial^{t_j} \left(\sum_{\alpha \in S} \phi^{v^\alpha} K_{[u_\alpha, g_j]} + k \sum_{\alpha \in S} \partial \phi^{v^\alpha} (u_\alpha, g) \right) \\
(3.20) \quad &= \sum_{\alpha \in S} \sum_{p=0}^{t_j} \binom{t_j}{p} \partial^p \phi^{v^\alpha} \partial^{t_j-p} K_{[u_\alpha, g_j]} + k \sum_{\alpha \in S} \partial^{t_j+1} \phi^{v^\alpha} (u_\alpha, g_j).
\end{aligned}$$

On the other hand, $\partial^T G = \bar{i}(\partial^T K_G)$ and we have
(3.21)

$$\{u_\alpha \lambda \partial^T G\} = \sum_{j=1}^t (\partial + \lambda)^{t_j} ([u_\alpha, g_j] + k\lambda(u_\alpha, g_j)) \partial^{t_1} g_1 \cdots \partial^{t_{j-1}} g_{j-1} \partial^{t_{j+1}} g_{j+1} \cdots \partial^{t_m} g_m.$$

Here,

$$(3.22) \quad (\partial + \lambda)^{t_j} ([u_\alpha, g_j] + k\lambda(u_\alpha, g_j)) = \sum_{p=0}^{t_j} \binom{t_j}{p} \lambda^p \partial^{t_j-p} [u_\alpha, g_j] + k\lambda^{t_j+1} (u_\alpha, g_j).$$

Let us assume that $X \in H(R_-, d_-)$ and $Z_{i,\alpha} \in S[\partial^j K_a | j \in \mathbb{Z}_{\geq 0}, a \in \bigoplus_{i \leq 1} \mathfrak{g}(i)]$ satisfy the equation: $d_-(X) = \sum_{\alpha \in S, i \in \mathbb{Z}_{\geq 0}} Z_{i,\alpha} \partial^i \phi^{v^\alpha}$. Then, by (3.19)-(3.22), the image $z_{i,\alpha}$ of $Z_{i,\alpha}$ under \bar{i} satisfies the equation $\{u_\alpha \lambda \bar{i}(X)\} = \sum_{i \in \mathbb{Z}_{\geq 0}} z_{i,\alpha} \lambda^i$. Hence we conclude that $[d_-(X) = 0 \Leftrightarrow Z_{i,\alpha} = 0 \text{ for any } i, \alpha \Leftrightarrow z_{i,\alpha} = 0 \text{ for any } i, \alpha \Leftrightarrow \{\mathfrak{n} \lambda \bar{i}(X)\} = 0]$.

To prove the second part of the theorem, let us pick two elements

$$\sum_{i \in I} \partial^{R_i} J_{A_i} \partial^{S_i} \Phi_{N_i}, \sum_{i' \in I'} \partial^{R_{i'}} J_{A_{i'}} \partial^{S_{i'}} \Phi_{N_{i'}} \in H(S(R_-), d_-).$$

Then the images of them under j are $\sum_{i \in I} (-1)^{l_i} \partial^{R_i} A_i \partial^{S_i} N_i$, and $\sum_{i' \in I'} (-1)^{l_{i'}} \partial^{R_{i'}} A_{i'} \partial^{S_{i'}} N_{i'}$ in $(S(\mathbb{C}[\partial] \otimes \mathfrak{g}) / \langle m - \chi(m) \rangle)^{\text{ad} \lambda^n}$. We want to show that

$$\begin{aligned}
(3.23) \quad & j \left(\left\{ \sum_{i \in I} \partial^{R_i} J_{A_i} \partial^{S_i} \Phi_{N_i} \lambda \sum_{i' \in I'} \partial^{R_{i'}} J_{A_{i'}} \partial^{S_{i'}} \Phi_{N_{i'}} \right\} \right) \\
&= \left\{ \sum_{i \in I} (-1)^{l_i} \partial^{R_i} A_i \partial^{S_i} N_i \lambda \sum_{i' \in I'} (-1)^{l_{i'}} \partial^{R_{i'}} A_{i'} \partial^{S_{i'}} N_{i'} \right\}.
\end{aligned}$$

Since $\{J_a \lambda \Phi_{[n]}\} = 0$, we have

$$\begin{aligned}
(3.24) \quad & \left\{ \sum_{i \in I} \partial^{R_i} J_{A_i} \partial^{S_i} \Phi_{N_i} \lambda \sum_{i' \in I'} \partial^{R_{i'}} J_{A_{i'}} \partial^{S_{i'}} \Phi_{N_{i'}} \right\} \\
&= \sum_{i \in I, i' \in I'} \partial^{S_{i'}} \Phi_{N_{i'}} \{ \partial^{R_i} J_{A_i} \lambda + \partial \partial^{R_{i'}} J_{A_{i'}} \} \rightarrow \partial^{S_i} \Phi_{N_i} \\
&+ \sum_{i \in I, i' \in I'} \partial^{R_{i'}} J_{A_{i'}} \{ \partial^{S_i} \Phi_{N_i} \lambda + \partial \partial^{S_{i'}} \Phi_{N_{i'}} \} \rightarrow \partial^{R_i} J_{A_i}.
\end{aligned}$$

Using the following two equalities:

$$j(\{J_a \lambda J_b\}) = i(J_{[a,b]} + \lambda k(a, b)) = \{a \lambda b\} \text{ for } a, b \in \bigoplus_{i \leq 0} \mathfrak{g}(i);$$

$$j(\{\Phi_{n_1} \lambda \Phi_{n_2}\}) = (f, [n_1, n_2]) = -\{n_1 \lambda n_2\} \text{ in } S(\mathbb{C}[\partial] \otimes \mathfrak{g}) / \langle m - \chi(m) \rangle, \text{ for } n_1, n_2 \in \mathfrak{g}\left(\frac{1}{2}\right);$$

we get the equation

$$(3.25) \quad j \left(\left\{ \sum_{i \in I} \partial^{R_i} J_{A_i} \partial^{S_i} \Phi_{N_i} \lambda \sum_{i' \in I'} \partial^{R_{i'}} J_{A_{i'}} \partial^{S_{i'}} \Phi_{N_{i'}} \right\} \right) \\ = (-1)^{l_i + l_{i'}} \left(\sum_{i \in I, i' \in I'} \partial^{S_{i'}} N_{i'} \{ \partial^{R_i} A_i \lambda + \partial \partial^{R_{i'}} A_{i'} \} \rightarrow \partial^{S_i} N_i \right. \\ \left. - \sum_{i \in I, i' \in I'} \partial^{R_{i'}} A_{i'} \{ \partial^{S_i} N_i \lambda + \partial \partial^{S_{i'}} N_{i'} \} \rightarrow \partial^{R_i} A_i \right).$$

On the other hand, the RHS of (3.23) can be rewritten as follows.

$$(3.26) \quad \left\{ \sum_{i \in I} \partial^{R_i} A_i \partial^{S_i} N_i \lambda \sum_{i' \in I'} \partial^{R_{i'}} A_{i'} \partial^{S_{i'}} N_{i'} \right\} = \sum_{i' \in I'} \partial^{S_{i'}} N_{i'} \left\{ \sum_{i \in I} \partial^{R_i} A_i \partial^{S_i} N_i \lambda \partial^{R_{i'}} A_{i'} \right\} \\ = \sum_{i \in I, i' \in I'} \partial^{S_{i'}} N_{i'} \{ \partial^{R_i} A_i \lambda + \partial \partial^{R_{i'}} A_{i'} \} \rightarrow \partial^{S_i} N_i - \sum_{i \in I, i' \in I'} \partial^{R_{i'}} A_{i'} \{ \partial^{S_i} N_i \lambda + \partial \partial^{S_{i'}} N_{i'} \} \rightarrow \partial^{R_i} A_i.$$

Here, we used the fact that $\{\mathfrak{n} \lambda \sum_{i \in I} \partial^{R_i} A_i \partial^{S_i} N_i\} = \{\mathfrak{n} \lambda \sum_{i' \in I'} \partial^{R_{i'}} A_{i'} \partial^{S_{i'}} N_{i'}\} = 0$. Hence (3.23) follows from (3.25) and (3.26) and hence the map j defined in (3.16) is a Poisson vertex algebra isomorphism between the two definitions of classical affine \mathcal{W} -algebras. \square

4. TWO EQUIVALENT DEFINITIONS OF CLASSICAL AFFINE FRACTIONAL \mathcal{W} -ALGEBRAS

We introduce notations (4.1)-(4.12) which are used in the rest of the paper.

Let us denote $\hat{\mathfrak{g}} := \mathfrak{g}[[z]]z \oplus \mathfrak{g}[-z]$. The Lie bracket and the symmetric bilinear form on $\hat{\mathfrak{g}}$ are defined by

$$(4.1) \quad [gz^i, hz^j] := [g, h]z^{i+j}, \quad (gz^i, hz^j) := (g, h)\delta_{i+j, 0}, \quad g, h \in \mathfrak{g}.$$

Also, we consider two different gradations gr_1 and gr_2 on $\hat{\mathfrak{g}}$ given by eigenvalues of the operations $\partial_1 = z\partial_z$ and $\partial_2 = (d+1)z\partial_z + \text{ad}_{\frac{h}{2}}$, where d is the largest eigenvalue of $\text{ad}_{\frac{h}{2}} : \mathfrak{g} \rightarrow \mathfrak{g}$. Then the two gradations have the following properties:

$$(4.2) \quad \text{gr}_1(z) = 1, \text{gr}_1(\mathfrak{g}) = 0, \text{gr}_2(z) = d+1, \text{gr}_2(g) = i \text{ for } g \in \mathfrak{g}_i.$$

If $n_1 \in \mathbb{Z}$ and $n_2 \in \frac{\mathbb{Z}}{2}$, then we denote subspaces of $\hat{\mathfrak{g}}$ as follows:

$$(4.3) \quad \hat{\mathfrak{g}}^{n_1} := \mathfrak{g}z^{n_1}, \quad \hat{\mathfrak{g}}_{n_2} := \{g \in \hat{\mathfrak{g}} | \text{gr}_2(g) = n_2\}, \quad \hat{\mathfrak{g}}_{n_2}^{n_1} := \hat{\mathfrak{g}}^{n_1} \cap \hat{\mathfrak{g}}_{n_2}, \\ \hat{\mathfrak{g}}_{<i} := \bigoplus_{n_2 < i} \hat{\mathfrak{g}}_{n_2}, \quad \hat{\mathfrak{g}}_{>i} := \bigoplus_{n_2 > i} \hat{\mathfrak{g}}_{n_2}, \quad \hat{\mathfrak{g}}^{\leq i} := \bigoplus_{n_1 \leq i} \hat{\mathfrak{g}}^{n_1}, \quad \hat{\mathfrak{g}}_{n_2}, \quad \hat{\mathfrak{g}}^{\geq i} := \bigoplus_{n_1 \geq i} \hat{\mathfrak{g}}^{n_1}.$$

In particular, we have $pz^j \in \hat{\mathfrak{g}}_{(d+1)j+d}^j$ and $fz^j \in \hat{\mathfrak{g}}_{(d+1)j-1}^j$.

For a given integer $m \geq 0$, there are two quotient spaces and two subspaces of $\hat{\mathfrak{g}}$ which play important roles in the following sections. The four vector spaces are denoted by:

$$(4.4) \quad \begin{aligned} \hat{\mathfrak{g}}_{[d,m]}^+ &:= \hat{\mathfrak{g}}^{\geq 0} / \hat{\mathfrak{g}}_{>(d+1)m+1}, & \hat{\mathfrak{g}}_{[d,m]}^- &:= \hat{\mathfrak{g}}^{\leq 0} / \hat{\mathfrak{g}}_{<-(d+1)m-1}, \\ \hat{\mathfrak{g}}_{<(d+1)m+1}^{\geq 0} &:= \hat{\mathfrak{g}}^{\geq 0} \cap \hat{\mathfrak{g}}_{<(d+1)m+1}, & \hat{\mathfrak{g}}_{>-(d+1)m-1}^{\leq 0} &:= \hat{\mathfrak{g}}^{\leq 0} \cap \hat{\mathfrak{g}}_{>-(d+1)m-1}. \end{aligned}$$

We notice that two subspaces $\hat{\mathfrak{g}}_{<(d+1)m+1}^{\geq 0}$ and $\hat{\mathfrak{g}}_{>-(d+1)m-1}^{\leq 0}$ are dual with respect to the bilinear form (\cdot, \cdot) on $\hat{\mathfrak{g}}$. Also, there are vector space isomorphisms $\iota : \hat{\mathfrak{g}}_{<(d+1)m+1}^{\geq 0} \oplus \hat{\mathfrak{g}}_{(d+1)m+1} \simeq \hat{\mathfrak{g}}_{[d,m]}^+$ and $\iota^- : \hat{\mathfrak{g}}_{>-(d+1)m-1}^{\leq 0} \oplus \hat{\mathfrak{g}}_{-(d+1)m-1} \simeq \hat{\mathfrak{g}}_{[d,m]}^-$ such that $\iota(a) = \bar{a}$ and $\iota^-(b) = \bar{b}$.

Let us denote

$$(4.5) \quad \Lambda_m = -fz^{-m} - pz^{-m-1} \in \hat{\mathfrak{g}}_{-(d+1)m-1}$$

and let χ be an element in $(\hat{\mathfrak{g}}_{(d+1)m+1})^*$ defined by

$$(4.6) \quad \chi(a) = (\Lambda_m, a), \quad a \in \hat{\mathfrak{g}}_{(d+1)m+1}.$$

In this section, we mainly deal with a differential algebra

$$(4.7) \quad \mathcal{V}_{[d,m]} := S(\mathbb{C}[\partial] \otimes \hat{\mathfrak{g}}_{[d,m]}^+) / I,$$

where I is the differential algebra ideal of $S(\mathbb{C}[\partial] \otimes \hat{\mathfrak{g}}_{[d,m]}^+)$ generated by $\{\iota(a) - \chi(a) | a \in \hat{\mathfrak{g}}_{(d+1)m+1}\}$. Then $S(\mathbb{C}[\partial] \otimes \hat{\mathfrak{g}}_{<(d+1)m+1}^{\geq 0})$ and $\mathcal{V}_{[d,m]}$ are isomorphic as differential algebras. Indeed, the differential algebra isomorphism

$$(4.8) \quad \tilde{\iota} : S(\mathbb{C}[\partial] \otimes \hat{\mathfrak{g}}_{<(d+1)m+1}^{\geq 0}) \simeq \mathcal{V}_{[d,m]}$$

induced by ι . Also, the Lie algebra $\hat{\mathfrak{g}} \otimes \mathcal{V}_{[d,m]}$ is endowed with the bracket and the bilinear form defined by

$$(4.9) \quad [a \otimes f, b \otimes g] = [a, b] \otimes fg \text{ and } (a \otimes f, b \otimes g) = (a, b)fg.$$

Let $\{u_i \mid i \in \bar{S}\}$ and $\{\tilde{u}_i \mid i \in \bar{S}\}$ be dual bases of \mathfrak{g} with respect to the bilinear form (\cdot, \cdot) and assume that the basis elements u_i and \tilde{u}_i are eigenvectors of $\text{ad}_{\frac{h}{2}}$. We denote $u_i^j := u_i z^j \in \hat{\mathfrak{g}}$ and $\tilde{u}_i^j := \tilde{u}_i z^j \in \hat{\mathfrak{g}}$ for any $j \in \mathbb{Z}$. Then $\{u_i^j \mid i \in \bar{S}, j \in \mathbb{Z}\}$ and $\{\tilde{u}_i^{-j} \mid i \in \bar{S}, j \in \mathbb{Z}\}$ are dual bases of $\hat{\mathfrak{g}}$. Also, the following two sets

$$(4.10) \quad \mathcal{B} := \left\{ u_i^j \mid i \in \bar{S}, j \in \mathbb{Z} \right\} \cap \hat{\mathfrak{g}}_{<(d+1)m+1}^{\geq 0} \text{ and } \mathcal{B}^- := \left\{ \tilde{u}_i^{-j} \mid i \in \bar{S}, j \in \mathbb{Z} \right\} \cap \hat{\mathfrak{g}}_{>-(d+1)m-1}^{\leq 0}$$

are bases of $\hat{\mathfrak{g}}_{<(d+1)m+1}^{\geq 0}$ and $\hat{\mathfrak{g}}_{>-(d+1)m-1}^{\leq 0}$ and these two bases are dual with respect to (\cdot, \cdot) .

The basis $\bar{\mathcal{B}}$ of $\hat{\mathfrak{g}}_{<(d+1)m+1}^{\geq 0} \oplus \hat{\mathfrak{g}}$ is an extended basis of \mathcal{B} such that

$$(4.11) \quad \bar{\mathcal{B}} := \mathcal{B} \cup \mathcal{B}_m, \text{ where } \mathcal{B}_m \text{ is a basis of } \hat{\mathfrak{g}}_{(d+1)m+1}.$$

The index sets \mathcal{I} , \mathcal{I}^- , and $\bar{\mathcal{I}}$ are defined by

$$(4.12) \quad \mathcal{I} = \{ (i, j) \mid u_i^j \in \mathcal{B} \}, \quad \mathcal{I}^- = \{ (i, j) \mid u_i^j \in \mathcal{B}^- \}, \quad \bar{\mathcal{I}} = \{ (i, j) \mid u_i^j \in \bar{\mathcal{B}} \}.$$

Using the isomorphism $\tilde{\iota}$, we often write an element in $\mathcal{V}_{[d,m]}$ as a polynomial in $\mathbb{C}[\partial^n \mathcal{B} \mid n \geq 0]$, by an abuse of notation.

4.1. First definition of classical affine fractional \mathcal{W} -algebras. In this section, we review the construction of classical affine fractional \mathcal{W} -algebras introduced in [1, 3]. Let \mathcal{F} be the algebra of complex valued smooth functions on S^1 and let $\{ \partial_x^n u_i^j(x) \in \mathcal{F} \mid n \in \mathbb{Z}_{\geq 0}, i \in \tilde{S}, j \in \mathbb{Z} \}$ be an algebraically independent set. We have a differential algebra homomorphism

$$\mu : \mathbb{C}_{\text{diff}}[\mathcal{B}] \rightarrow \mathcal{F},$$

such that $\mu : u_i^{j(n)} \mapsto \frac{\partial_x^n u_i^j(x)}{\partial x^n}$ if $u_i^j \in \mathcal{B}$ and $\mu : u_i^{j(n)} \mapsto \delta_{n,0} \cdot (\Lambda_m, u_i^j)$ if $u_i^j \in \mathcal{B}_m$. Then $\mathcal{V}_{[d,m]}$ is isomorphic to the image $\mu(\mathcal{V}_{[d,m]})$. Hence functions in $\mu(\mathcal{V}_{[d,m]})$ can be identified with elements in $\mathcal{V}_{[d,m]}$.

When we need to clarify that a function $f \in \mathcal{V}_{[d,m]}$ depends on a variable $x \in S^1$, we denote f by $f(x)$. Especially, if there are more than one independent variables in a formula, we indicate the variable of each functional.

Definition 4.1. For given

$$(4.13) \quad q_m \in \hat{\mathfrak{g}}_{>-(d+1)m-1}^{\leq 0} \otimes \mathcal{V}_{[d,m]} \text{ and } k \in \mathbb{C},$$

an operator of the form

$$(4.14) \quad L_m(x) = k\partial + q_m(x) + \Lambda_m \otimes 1 \in \mathbb{C}\partial \ltimes \hat{\mathfrak{g}} \otimes \mathcal{V}_{[d,m]}$$

is called the Lax operator associated to q_m .

The differential ∂ acts on $\hat{\mathfrak{g}} \otimes \mathcal{V}_{[d,m]}$ by

$$(4.15) \quad \partial(a \otimes f(x)) := a \otimes (\partial f(x)).$$

Then one can check that $\partial[a \otimes f(x), b \otimes g(x)] = [a \otimes \partial f(x), b \otimes g(x)] + [a \otimes f(x) + b \otimes \partial g(x)]$. The Lax operator $L_m = k\partial + \sum_{(i,j) \in \mathcal{I}} \tilde{u}_i^{-j} \otimes q_m^{i,j} + \Lambda_m \otimes 1$ linearly acts on $\hat{\mathfrak{g}} \otimes \mathcal{V}_{[d,m]}$ by the adjoint action:

$$[L_m(x), a \otimes f(x)] = ka \otimes \partial f(x) + \sum_{(i,j) \in \mathcal{I}} [\tilde{u}_i^{-j}, a] \otimes q_m^{i,j}(x)f(x) + [\Lambda_m, a] \otimes f(x).$$

Recall that we have a bilinear form (\cdot, \cdot) on $\hat{\mathfrak{g}} \otimes \mathcal{V}_{[d,m]}$ defined by $(a \otimes f, b \otimes g) = (a, b)fg$.

Definition 4.2. Let $a, b \in \hat{\mathfrak{g}}$, $x, y, w \in S^1$ and $g_i \in \mathcal{F}$ for $i = 1, 2, 3, 4$. A bilinear form $(\cdot, \cdot)_w$ is defined by

$$(4.16) \quad (a \otimes F(x, w), b \otimes G(y, w))_w := \int_{S^1} F(x, w)G(y, w) dw,$$

where $F(x, w) = g_1(x)g_2(w)\partial_x^n \delta(x-w)$, or $g_1(x)g_2(w)$ and $G(y, w) = g_3(y)g_4(w)\partial_y^m \delta(y-w)$ or $g_3(y)g_4(w)$.

Using the bilinear form, a basis element $u_i^j \in \mathcal{B}$ can be understood as a functional on $\hat{\mathfrak{g}}_{>-(d+1)m-1}^{\leq 0} \otimes \mathcal{V}_{[d,m]}$, i.e.

$$(4.17) \quad u_i^j(a \otimes f(x)) := (u_i^j \otimes 1, a \otimes f(x)) = (u_i^j \otimes \delta(x-w), a \otimes f(w))_w,$$

where $a \in \hat{\mathfrak{g}}_{>-(d+1)m-1}^{\leq 0}$ and $f \in \mathcal{V}_{[d,m]}$. Moreover, we let

$$(4.18) \quad (\partial u_i^j)(a \otimes f) = \partial(u_i^j(a \otimes f)), \quad (g_1 g_2)(a \otimes f) = (g_1(a \otimes f)) \cdot (g_2(a \otimes f)), \quad c(a \otimes f) = c,$$

for $g_1, g_2 \in \mathcal{V}_{[d,m]}$ and $c \in \mathbb{C}$. Hence every element in $\mathcal{V}_{[d,m]}$ is a functional on $\hat{\mathfrak{g}} \otimes \mathcal{V}_{[d,m]}$.

Remark 4.3. For $f, g \in \mathcal{V}_{[d,m]}$ and $a \in \hat{\mathfrak{g}}$, the functional multiplication is defined by $g \cdot (a \otimes f) = a \otimes gf$.

Lemma 4.4. For any $Q_m \in \hat{\mathfrak{g}}_{>-(d+1)m-1}^{\leq 0} \otimes \mathcal{V}_{[d,m]}$ and $q_m^{i,j} \in \mathcal{V}_{[d,m]}$, the operator

$$L_m(Q_m) := k\partial + \sum_{(i,j) \in \mathcal{I}} \tilde{u}_i^{-j} \otimes (q_m^{i,j}(Q_m)) + \Lambda_m \otimes 1$$

is a Lax operator.

Lemma 4.4 follows from the fact that $q_m^{i,j}$ is a functional on $\hat{\mathfrak{g}}_{>-(d+1)m-1}^{\leq 0} \otimes \mathcal{V}_{[d,m]}$. Also, the operator

$$(4.19) \quad \mathcal{L}_m := k\partial + \sum_{(i,j) \in \mathcal{I}} \tilde{u}_i^{-j} \otimes u_i^j + \Lambda_m \otimes 1 = k\partial + \sum_{(i,j) \in \mathcal{I}} \tilde{u}_i^{-j} \otimes u_i^j \in \mathbb{C}\partial \ltimes \hat{\mathfrak{g}} \otimes \mathcal{V}_{[d,m]}$$

is called the universal Lax operator. Then a Lax operator L_m has the form

$$(4.20) \quad L_m = k\partial + q_m + \Lambda_m \otimes 1 = \mathcal{L}_m(q_m).$$

Now we introduce the gauge equivalence class on $\hat{\mathfrak{g}} \otimes \mathcal{V}_{[d,m]}$.

Definition 4.5. Let $S = a \otimes f \in \hat{\mathfrak{g}} \otimes \mathcal{V}_{[d,m]}$ and $b \otimes g \in \hat{\mathfrak{g}} \otimes \mathcal{V}_{[d,m]}$. The adjoint action of the Lie algebra $\hat{\mathfrak{g}} \otimes \mathcal{V}_{[d,m]}$ on the space $\mathbb{C}\partial \ltimes \hat{\mathfrak{g}} \otimes \mathcal{V}_{[d,m]}$ is defined as follows:

$$(\text{ad} S)(\partial) = -a \otimes \partial f, \text{ and } (\text{ad} S)(b \otimes g) = [a, b] \otimes fg.$$

Definition 4.6. Let $S \in \mathfrak{n} \otimes \mathcal{V}_{[d,m]}$ and consider the map

$$(4.21) \quad L_m = k\partial + q_m + \Lambda_m \otimes 1 \mapsto \widetilde{L}_m := e^{\text{ad} S}(L_m) = k\partial + \widetilde{q}_m + \Lambda_m \otimes 1$$

between Lax operators. Then the map $G_S : \hat{\mathfrak{g}} \otimes \mathcal{V}_{[d,m]} \rightarrow \hat{\mathfrak{g}} \otimes \mathcal{V}_{[d,m]}$ such that $G_S : q_m \mapsto \widetilde{q}_m$ is called the gauge transformation by S . Since gauge transformations define an equivalence relation, we write $q_m \sim \widetilde{q}_m$ or $q_m \sim_S \widetilde{q}_m$ if $\widetilde{q}_m = G_S(q_m)$.

Any element in the differential algebra $\mathcal{V}_{[d,m]}$ is a map from $\hat{\mathfrak{g}}_{>-(d+1)m-1}^{\leq 0} \otimes \mathcal{V}_{[d,m]}$ to $\mathcal{V}_{[d,m]}$, as described in (4.17). Instead of the whole algebra $\mathcal{V}_{[d,m]}$, we focus on the subset of $\mathcal{V}_{[d,m]}$ consisting of functionals which are well-defined on the gauge equivalence classes of $\hat{\mathfrak{g}}_{>-(d+1)m-1}^{\leq 0} \otimes \mathcal{V}_{[d,m]}$.

Definition 4.7. A functional $a : \hat{\mathfrak{g}}_{>-(d+1)m-1}^{\leq 0} \otimes \mathcal{V}_{[d,m]} \rightarrow \mathcal{V}_{[d,m]}$ is said to be gauge invariant if $a(q_m) = a(\tilde{q}_m)$ whenever q_m and \tilde{q}_m are gauge equivalent.

Definition 4.8. The subset of $\mathcal{V}_{[d,m]}$ consisting of gauge invariant functionals on $\hat{\mathfrak{g}}_{>-(d+1)m-1}^{\leq 0} \otimes \mathcal{V}_{[d,m]}$ is called the m -th affine fractional \mathcal{W} -algebra associated to \mathfrak{g} , Λ_m and k and we write this algebra as

$$(4.22) \quad \mathcal{W}_1(\mathfrak{g}, \Lambda_m, k).$$

Affine fractional \mathcal{W} -algebras are well-defined differential algebras since if ϕ and ψ are gauge invariant then $\phi + \psi$, $\phi\psi$ and $\partial\phi$ are also gauge invariant. In the rest of this section, we introduce two local Poisson brackets on each affine fractional \mathcal{W} -algebra.

Remark 4.9. In Section 4.2, we introduce another definition of affine fractional \mathcal{W} -algebras. The subscript 1 in (4.22) is used until we prove the equivalence of two definitions.

Definition 4.10. Let $\phi, \psi \in \mathcal{W}_1(\mathfrak{g}, \Lambda_m, k)$. Then the two local brackets $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ on $\mathcal{W}_1(\mathfrak{g}, \Lambda_m, k)$ are defined by

$$(4.23) \quad \{\phi(x), \psi(y)\}_1 = - \left(\sum_{\substack{(i,j) \in \mathcal{I}, \\ n \geq 0}} u_i^j \otimes \frac{\partial \phi(x)}{\partial u_i^{j(n)}} \partial_x^n \delta(x-w), \left[\sum_{\substack{(p,q) \in \mathcal{I}, \\ l \geq 0}} u_p^{q+1} \otimes \frac{\partial \psi(y)}{\partial u_p^{q(l)}} \partial_y^l \delta(y-w), \mathcal{L}_m(w) \right] \right)_w,$$

$$(4.24) \quad \{\phi(x), \psi(y)\}_2 = \left(\sum_{\substack{(i,0) \in \mathcal{I}, \\ n \geq 0}} u_i^0 \otimes \frac{\partial \phi(x)}{\partial u_i^{0(n)}} \partial_x^n \delta(x-w), \left[\sum_{\substack{(p,0) \in \mathcal{I}, \\ l \geq 0}} u_p^{0(l)} \otimes \frac{\partial \psi(y)}{\partial u_p^{0(l)}} \partial_y^l \delta(y-w), \mathcal{L}_m(w) \right] \right)_w \\ - \left(\sum_{\substack{(i,j) \in \mathcal{I}, j > 0, \\ n \geq 0}} u_i^j \otimes \frac{\partial \phi(x)}{\partial u_i^{j(n)}} \partial_x^n \delta(x-w), \left[\sum_{\substack{(p,q) \in \mathcal{I}, \\ q > 0, l \geq 0}} u_p^q \otimes \frac{\partial \psi(y)}{\partial u_p^{q(l)}} \partial_y^l \delta(y-w), \mathcal{L}_m(w) \right] \right)_w.$$

(Later, in Proposition 4.14, we show that $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ are well-defined Poisson local brackets.)

Remark 4.11. The local Poisson brackets $\{\phi(x), \psi(y)\}_i$, $i = 1, 2$, have the form $\sum_{n \geq 0} \Phi_{i,n}(y) \partial_y^n \delta(x-y)$, for some $\Phi_{i,n} \in \mathcal{V}_{[d,m]}$. By definition of δ -function, we have $\int \Phi_{i,n}(y) \partial_y^n \delta(x-y) dx = (-\partial)^n \Phi_{i,n}(y)$. Hence Poisson brackets on $\mathcal{V}_{[d,m]}/\partial \mathcal{V}_{[d,m]}$ can be defined by

$$\int \{\phi, \psi\}_i := \int \int \{\phi(x), \psi(y)\}_i dx dy.$$

Two brackets $\int \{\cdot, \cdot\}_i$, $i = 1, 2$, are well-defined Poisson brackets on $\mathcal{W}_1(\mathfrak{g}, \Lambda_m, k)/\partial\mathcal{W}_1(\mathfrak{g}, \Lambda_m, k)$ if the local brackets $\{\cdot, \cdot\}_i$ are well-defined on $\mathcal{W}_1(\mathfrak{g}, \Lambda_m, k)$. Indeed, the skew-commutativity, Jacobi identity and Leibniz rules of $\{\cdot, \cdot\}_i$ guarantee those of $\int \{\cdot, \cdot\}_i$.

Lemma 4.12. *The derivative of ϕ at $q_m(x)$ has the following property:*

$$(4.25) \quad \frac{d}{d\epsilon} \phi(q_m(x) + \epsilon r(x))|_{\epsilon=0} = \sum_{\substack{(i,j) \in \mathcal{I}, \\ n \geq 0}} \frac{\partial \phi(q_m(x))}{\partial u_i^{j(n)}} (u_i^j \otimes \partial_x^n \delta(x-w), r(w))_w.$$

Proof. By Taylor expansion, we can write $\phi(q_m(x) + \epsilon r(x))$ as a power series in ϵ . Substituting $\epsilon = 0$ to the series, we obtain the following formula:

$$\begin{aligned} \frac{d}{d\epsilon} \phi(q_m(x) + \epsilon r(x))|_{\epsilon=0} &= \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \left(\phi(q_m(x)) + \epsilon \sum_{\substack{(i,j) \in \mathcal{I}, \\ n \in \mathbb{Z}_{\geq 0}}} \frac{\partial \phi}{\partial u_i^{j(n)}} \partial_x^n (u_i^j \otimes 1, r(x)) + o(\epsilon^2) \right) \\ &= \sum_{(i,j) \in \mathcal{I}, n \geq 0} \frac{\partial \phi(q_m(x))}{\partial u_i^{j(n)}} (u_i^j \otimes \partial_x^n \delta(x-w), r(w))_w. \end{aligned}$$

□

Lemma 4.13. *Let $L_m(x) = \partial + q_m(x) + \Lambda_m \otimes 1$ and let $e^{adS(x)} L_m(x) = \tilde{L}_m(x) = \partial + \tilde{q}_m(x) + \Lambda_m \otimes 1$. Then, for $\phi, \psi \in \mathcal{W}_1(\mathfrak{g}, \Lambda_m, k)$, we have*

$$(4.26) \quad \sum_{\substack{(i,j) \in \mathcal{I}, \\ n \geq 0}} u_i^j \otimes \frac{\partial \phi(\tilde{q}_m(x))}{\partial u_i^{j(n)}} \partial_x^n \delta(x-w) = \sum_{\substack{(i,j) \in \mathcal{I}, \\ n \geq 0}} \frac{\partial \phi(q_m(x))}{\partial u_i^{j(n)}} e^{adS(w)} (u_i^j \otimes \partial_x^n \delta(x-w)).$$

Proof. By Lemma 4.12, we have two equations:

$$\begin{aligned} \frac{d}{d\epsilon} \phi(\tilde{q}_m(x) + \epsilon r(x)) \bigg|_{\epsilon=0} &= \sum_{\substack{(i,j) \in \mathcal{I}, \\ n \geq 0}} \frac{\partial \phi(\tilde{q}_m(x))}{\partial u_i^{j(n)}} (u_i^j \otimes \partial_x^n \delta(x-w), r(w))_w; \\ \frac{d}{d\epsilon} \phi(q_m + e^{-adS} \epsilon r) \bigg|_{\epsilon=0} &= \sum_{\substack{(i,j) \in \mathcal{I}, \\ n \geq 0}} \frac{\partial \phi(q_m(x))}{\partial u_i^{j(n)}} (e^{adS(w)} (u_i^j \otimes \partial_x^n \delta(x-w)), r(w))_w. \end{aligned}$$

Since $q_m + e^{-adS} \epsilon r \sim_S \tilde{q}_m + \epsilon r$, we obtain (4.26). □

To simplify notations, let us denote

$$\begin{aligned} d_{q_m(x)} \phi(w) &:= \sum_{(i,j) \in \mathcal{I}, n \geq 0} u_i^j \otimes \frac{\partial \phi(q_m(x))}{\partial u_i^{j(n)}} \partial_x^n \delta(x-w), \\ d_{q_m(x)} \phi(w)^0 &:= \sum_{(i,0) \in \mathcal{I}, n \geq 0} u_i^0 \otimes \frac{\partial \phi(q_m(x))}{\partial u_i^{0(n)}} \partial_x^n \delta(x-w), \\ d_{q_m(x)} \phi(w)^> &:= \sum_{(i,j) \in \mathcal{I}, j > 0, n \geq 0} u_i^j \otimes \frac{\partial \phi(q_m(x))}{\partial u_i^{j(n)}} \partial_x^n \delta(x-w). \end{aligned}$$

Then the two brackets $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ are

$$\begin{aligned}\{\phi(x), \psi(y)\}_1 &= -(d_{q_m(x)}\phi(w), [zd_{q_m(y)}\psi(w), \mathcal{L}_m(w)])_w; \\ \{\phi(x), \psi(y)\}_2 &= (d_{q_m(x)}\phi(w)^0, [d_{q_m(y)}\psi(w)^0, \mathcal{L}_m(w)])_w - (d_{q_m(x)}\phi(w)^>, [d_{q_m(y)}\psi(w)^>, \mathcal{L}_m(w)])_w.\end{aligned}$$

Proposition 4.14. *Two local Poisson brackets $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ are well-defined on $\mathcal{W}_1(\mathfrak{g}, \Lambda_m, k)$.*

Proof. Recall that $q_m \sim_S \tilde{q}_m$ if and only if $e^{\text{ad}S}\mathcal{L}_m(q_m) = \mathcal{L}_m(\tilde{q}_m)$ and, by Lemma 4.13, $d_{\tilde{q}_m(x)}\phi(w) = e^{\text{ad}S(w)}d_{q_m(x)}\phi(w)$. Since the bilinear form $(\cdot, \cdot)_w$ is invariant under the adjoint action, we have

$$\begin{aligned}\{\phi(x), \psi(y)\}_1(\tilde{q}_m) &= -(e^{\text{ad}S(w)}d_{q_m(x)}\phi(w), [e^{\text{ad}S(w)}zd_{q_m(y)}\psi(w), e^{\text{ad}S(w)}\mathcal{L}_m(w)])_w \\ &= -(e^{\text{ad}S(w)}d_{q_m(x)}\phi(w), e^{\text{ad}S(w)}[zd_{q_m(y)}\psi(w), \mathcal{L}_m(w)])_w \\ &= -(d_{q_m(x)}\phi(w), [zd_{q_m(y)}\psi(w), \mathcal{L}_m(w)])_w = \{\phi(x), \psi(y)\}_1(q_m).\end{aligned}$$

Since $S \in \hat{\mathfrak{g}}^0 \otimes \mathcal{V}_{[d,m]}$, the same procedure works for the second bracket, i.e.

$$\{\phi(x), \psi(y)\}_2(\tilde{q}_m) = \{\phi(x), \psi(y)\}_2(q_m).$$

Hence $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ are well-defined on $\mathcal{W}_1(\mathfrak{g}, \Lambda_m, k)$.

To see the skew-symmetry, Jacobi identity and Leibniz rules, let us rewrite formulas (4.23) and (4.24). The first bracket is as follows:

$$\begin{aligned}\{\phi(x), \psi(y)\}_1 &= - \sum_{i,j,n} \sum_{p,q,l} \frac{\partial \phi(x)}{\partial u_i^{j(n)}} \partial_x^n \frac{\partial \psi(y)}{\partial u_p^{q(l)}} \partial_y^l [u_i^j, u_p^{p+1}](y) \delta(x-y).\end{aligned}$$

Similarly the second bracket of $\phi(x)$ and $\psi(y)$ can be written as follows:

$$\begin{aligned}\{\phi(x), \psi(y)\}_2 &= \sum_{u_i^0 \in \mathfrak{g}, n \geq 0} \sum_{u_p^0 \in \mathfrak{g}, l \geq 0} \frac{\partial \phi(x)}{\partial u_i^{0(n)}} \partial_x^n \frac{\partial \psi(y)}{\partial u_p^{0(l)}} \partial_y^l (k \partial_y(u_i^0, u_p^0) + [u_i^0, u_p^0](y)) \delta(x-y) \\ &\quad - \sum_{u_i^j \in S, j > 0, n \geq 0} \sum_{u_p^q \in S, q > 0, l \geq 0} \frac{\partial \phi(x)}{\partial u_i^{j(n)}} \partial_x^n \frac{\partial \psi(y)}{\partial u_p^{q(l)}} \partial_y^l [u_i^j, u_p^q](y) \delta(x-y).\end{aligned}$$

Then Leibniz rules hold obviously. Skew-symmetries and Jacobi identities of two brackets follow from those of $\hat{\mathfrak{g}}$ and the properties of δ -function. \square

The following theorem follows from Proposition 4.14 and Remark 4.11.

Theorem 4.15. (i) *The two brackets $\{\cdot, \cdot\}_i$, $i = 1, 2$, are well-defined local Poisson brackets on the affine fractional \mathcal{W} -algebra $\mathcal{W}(\mathfrak{g}, \Lambda_m, k)$.*
(ii) *The two brackets $\int \{\cdot, \cdot\}_i$ on the algebra $\mathcal{W}(\mathfrak{g}, \Lambda_m, k)/\partial \mathcal{W}(\mathfrak{g}, \Lambda_m, k)$, which are induced from $\{\cdot, \cdot\}_i$, are well-defined Poisson brackets.*

4.2. Second definition of classical affine fractional \mathcal{W} -algebras. In this section, we construct a PVA called a classical affine fractional \mathcal{W} -algebra which is equivalent to the fractional \mathcal{W} -algebra explained in the previous section.

Let λ -adjoint action $\text{ad}_\lambda \mathfrak{n}$ on $\mathcal{V}_{[d,m]}$ be defined as follows:

$$\begin{aligned} (\text{ad}_\lambda n)(az^i) &= \{n_\lambda az^i\}, \quad \text{where } \{n_\lambda az^i\} = [n, a] + \delta_{i,0} k \lambda(n, a), \\ (\text{ad}_\lambda n)(\partial A) &= (\partial + \lambda)(\text{ad}_\lambda n)(A), \quad (\text{ad}_\lambda n)(AB) = B(\text{ad}_\lambda n)(A) + A(\text{ad}_\lambda n)(B), \end{aligned}$$

for $a \in \mathfrak{g}$, $n \in \mathfrak{n}$, and $A, B \in \mathcal{V}_{[d,m]}$.

Definition 4.16. The m -th affine fractional \mathcal{W} -algebra associated to \mathfrak{g} , Λ_m and k is an associative differential algebra

$$(4.27) \quad \mathcal{W}_2(\mathfrak{g}, \Lambda_m, k) := \mathcal{V}_{[d,m]}^{\text{ad}_\lambda \mathfrak{n}}$$

endowed with the product $(a + I)(b + I) = ab + I$, where I is the differential algebra ideal of $S(\mathbb{C}[\partial] \otimes \hat{\mathfrak{g}}_{[d,m]}^+)$ such that $\mathcal{V}_{[d,m]} = S(\mathbb{C}[\partial] \otimes \hat{\mathfrak{g}}_{[d,m]}^+)/I$.

Proposition 4.17. *The associative differential algebra $\mathcal{W}_2(\mathfrak{g}, \Lambda_m, k)$ is well-defined.*

Proof. We need to show that $(\text{ad}_\lambda \mathfrak{n})(I) \subset I[\lambda]$. Let us pick $n \in \mathfrak{n}$, $A \in S(\mathbb{C}[\partial] \otimes \hat{\mathfrak{g}}_{(m)})$ and $B = a - \chi(a)$, $a \in \hat{\mathfrak{g}}_{(d+1)m+1}$. Then, by Leibniz rule,

$$\{n_\lambda AB\} = B(\text{ad}_\lambda n)(A) + A(\text{ad}_\lambda n)(B).$$

The first term of the RHS is clearly in I and the second term is also in I since $(\text{ad}_\lambda n)(B) = 0$. Hence $\text{ad}_\lambda \mathfrak{n}$ -action is well-defined on $S(\mathbb{C}[\partial] \otimes \hat{\mathfrak{g}}_{(m)})/I$. Also, since λ -adjoint action satisfies Leibniz rule, the product on $\mathcal{W}_2(\mathfrak{g}, \Lambda_m, k)$ is well-defined and, since $(\text{ad}_\lambda n)(\partial A) = (\partial + \lambda)((\text{ad}_\lambda n)(A))$, we conclude that $\mathcal{W}_2(\mathfrak{g}, \Lambda_m, k)$ is a well-defined differential associative algebra. \square

We want to show that that $\mathcal{W}_1(\mathfrak{g}, \Lambda_m, k)$ and $\mathcal{W}_2(\mathfrak{g}, \Lambda_m, k)$ are isomorphic as differential associative algebras.

Proposition 4.18. *Let $Q_m = \sum_{(i,j) \in \mathcal{I}} \tilde{u}_i^{-j} \otimes u_i^j \in \hat{\mathfrak{g}}_{(m)}^- \otimes \mathcal{V}_{[d,m]}$. Then the universal Lax operator $\mathcal{L}_m = k\partial + Q_m$. Also, for a given $S \in \mathfrak{n} \otimes \mathcal{V}_{[d,m]}$, we denote*

$$\mathcal{L}_m(\epsilon) = k\partial + Q_m(\epsilon) = e^{\text{ad } \epsilon S} \mathcal{L}_m.$$

If $S = a \otimes r$, then the derivative of $\phi \in \mathcal{V}_{[d,m]}$ has the following property:

$$(4.28) \quad \frac{d}{d\epsilon} \phi(Q_m(\epsilon))|_{\epsilon=0} = -\{a_\partial \phi\}_{k \rightarrow r}.$$

Proof. Since $\mathcal{L}_m(\epsilon) = \mathcal{L}_m + \text{ad } \epsilon S(\mathcal{L}_m) + o(\epsilon^2)$, we can write $Q_m(\epsilon)$ as a power series of ϵ :

$$Q_m(\epsilon) = Q_m + \text{ad } \epsilon S(\mathcal{L}_m) + o(\epsilon^2).$$

By Taylor expansion, we have

$$(4.29) \quad \phi(Q_m(\epsilon)) = \phi(Q_m) + \epsilon \sum_{\substack{(i,j) \in \mathcal{I}, n \geq 0 \\ (\alpha, \beta) \in \mathcal{I}}} \frac{\partial \phi(Q_m)}{\partial u_i^{j(n)}} \partial^n (-ka \otimes \partial r + [a \otimes r, \tilde{u}_\alpha^{-\beta} \otimes u_\alpha^\beta], u_i^j \otimes \delta(x-w))_w + o(\epsilon^2).$$

The second term of the RHS can be rewritten as

$$\begin{aligned}
& ([a \otimes r, k\partial + \sum_{(\alpha, \beta) \in \mathcal{I}} \tilde{u}_\alpha^{-\beta} \otimes u_\alpha^\beta], u_i^j \otimes \delta(x-w))_w \\
&= (-ka \otimes \partial r + \sum_{(\alpha, \beta) \in \mathcal{I}} [a, \tilde{u}_\alpha^{-\beta}] \otimes ru_\alpha^\beta, u_i^j \otimes \delta(x-w))_w = -[a, u_i^j]r - k(a, u_i^j)\partial r.
\end{aligned}$$

We proved (4.28), since

$$\{a_\partial \phi\}_{k \rightarrow r} = \sum_{\substack{(i,j) \in \mathcal{I}, \\ n \geq 0}} \frac{\partial \phi}{\partial u_i^{j(n)}} \partial^n \{a_\partial u_i^j\}_{\rightarrow r} = \sum_{\substack{(i,j) \in \mathcal{I}, \\ n \geq 0}} \frac{\partial \phi}{\partial u_i^{j(n)}} \partial^n ([a, u_i^j]r + k(a, u_i^j)\partial r).$$

□

Corollary 4.19. *Two differential algebras $\mathcal{W}_1(\mathfrak{g}, \Lambda_m, k)$ and $\mathcal{W}_2(\mathfrak{g}, \Lambda_m, k)$ are isomorphic. Hence we denote the m -th affine fractional \mathcal{W} -algebra associated to \mathfrak{g} , Λ_m and k by $\mathcal{W}(\mathfrak{g}, \Lambda_m, k)$.*

Proof. Let $Q_m = \sum_{(i,j) \in \bar{\mathcal{I}}} \tilde{u}_i^{-j} \otimes u_i^j$ and let $\mathcal{L}_m(\epsilon) = k\partial + Q_m(\epsilon) = e^{\text{ad} \epsilon S} \mathcal{L}_m$, where $S = a \otimes f \in \mathfrak{n} \otimes \mathcal{V}_{[d,m]}$. By the definition of $\mathcal{W}_1(\mathfrak{g}, \Lambda_m, k)$, if a functional ϕ is in $\mathcal{W}_1(\mathfrak{g}, \Lambda, k)$ then $\frac{d}{d\epsilon} \phi(Q_m(\epsilon))|_{\epsilon=0} = 0$. Notice that

$$[\{n_\lambda \phi\} = 0 \text{ for any } n \in \mathfrak{n}] \text{ if and only if } [\{n_\partial \phi\}_{\rightarrow r} = 0 \text{ for any } a \otimes r \in \mathfrak{n} \otimes \mathcal{V}_{[d,m]}].$$

Thus, by Proposition 4.18, if $\frac{d}{d\epsilon} \phi(Q_m(\epsilon))|_{\epsilon=0} = 0$, then a functional ϕ is in $\mathcal{W}_2(\mathfrak{g}, \Lambda_m, k)$. Hence $\mathcal{W}_1(\mathfrak{g}, \Lambda, k) \subset \mathcal{W}_2(\mathfrak{g}, \Lambda, k)$.

Conversely, assume that $\phi \in \mathcal{W}_2(\mathfrak{g}, \Lambda_m, k)$. Then, by Proposition 4.18, $\frac{d}{d\epsilon} \phi(Q_m(\epsilon))|_{\epsilon=0} = 0$. Let $L_m := k\partial + q_m = \mathcal{L}(q_m) = k\partial + Q_m(q_m)$ be a Lax operator and let $L_m(\epsilon) := e^{\text{ad} \epsilon S} L_m = k\partial + q_m(\epsilon)$. If $S = a \otimes r \in \mathfrak{n} \otimes \mathcal{V}_{[d,m]}$, then

$$(4.30) \quad \left. \frac{d\phi(q_m(\epsilon))}{d\epsilon} \right|_{\epsilon=0} = ([a \otimes 1, k\partial + \sum_{\alpha, \beta} \tilde{u}_\alpha^{-\beta} \otimes u_\alpha^\beta], u_i^j \otimes \delta(x-w))_w \Big|_{u_i^j = (\tilde{u}_i^{-j}, q_m)} \cdot r.$$

Hence if

$$\left. \frac{d\phi(Q_m(\epsilon))}{d\epsilon} \right|_{\epsilon=0} = ([a \otimes 1, k\partial + \sum_{\alpha, \beta} \tilde{u}_\alpha^{-\beta} \otimes u_\alpha^\beta], u_i^j \otimes \delta(x-w))_w r = 0$$

for any $a \otimes r \in \mathfrak{n} \otimes \mathcal{V}_{[d,m]}$, then $\frac{d\phi(q_m(\epsilon))}{d\epsilon} \Big|_{\epsilon=0} = 0$ for any $a \otimes r \in \mathfrak{n} \otimes \mathcal{V}_{[d,m]}$ and the functional ϕ is gauge invariant, i.e. $\phi \in \mathcal{W}_1(\mathfrak{g}, \Lambda_m, k)$. □

Let us define Poisson λ -brackets on $\mathcal{W}(\mathfrak{g}, \Lambda_m, k)$ using the local brackets $\{\cdot, \cdot\}_i$, $i = 1, 2$. Suppose the local Poisson brackets are

$$(4.31) \quad \{\phi(x), \psi(y)\}_i =: \sum_{j \geq 0} \frac{1}{j!} (\phi_{(j)} \psi)_{(i)}(y) \partial_y^j \delta(x-y), \quad \phi, \psi \in \mathcal{W}(\mathfrak{g}, \Lambda_m, k), \quad i = 1, 2.$$

Then we can define corresponding Poisson λ -brackets by

$$(4.32) \quad \{\phi_\lambda \psi\}_i(y) := \int e^{\lambda(x-y)} \{\phi(x), \psi(y)\}_i dx = \sum_{j \geq 0} \frac{\lambda^j}{j!} (\phi_{(j)} \psi)_i(y), \quad i = 1, 2.$$

The algebra $\mathcal{W}(\mathfrak{g}, \Lambda_m, k)$ is a well-defined PVA endowed with the λ -brackets $\{\cdot_\lambda \cdot\}_i$, $i = 1, 2$. (see Proposition 4.21)

Lemma 4.20. *For any $\phi, \psi \in \mathcal{W}(\mathfrak{g}, \Lambda_m, k)$, we have*

$$(4.33) \quad \{\partial_x \phi(x), \psi(y)\}_i = \partial_x \{\phi(x), \psi(y)\}_i \text{ and } \{\phi(x), \partial_y \psi(y)\}_i = \partial_y \{\phi(x), \psi(y)\}_i.$$

Proof. By the product rule of derivatives, we obtain

$$(4.34) \quad \begin{aligned} \sum_{n \geq 0} \frac{\partial}{\partial u_i^{j(n)}} (\partial_x \phi(x)) \partial_x^n \delta(x - w) &= \sum_{n \geq 0} \left[\left(\partial_x \frac{\partial}{\partial u_i^{j(n)}} \phi(x) \right) \partial_x^n \delta(x - w) + \frac{\partial}{\partial u_i^{j(n-1)}} \phi(x) \partial_x^n \delta(x - w) \right] \\ &= \sum_{n \geq 0} \partial_x \left(\frac{\partial}{\partial u_i^{j(n)}} \phi \partial_x^n \delta(x - w) \right). \end{aligned}$$

Applying equation (4.34) to the brackets (4.23) and (4.24), we get the equation (4.33). \square

Proposition 4.21. *Two λ -brackets (4.32) are well-defined on $\mathcal{W}(\mathfrak{g}, \Lambda_m, k)$.*

Proof. For any $\phi, \psi \in \mathcal{W}(\mathfrak{g}, \Lambda_m, k)$, we already proved that $\{\phi(x), \psi(y)\}_i$ is gauge invariant. Hence by (4.32), we have $\{\phi_\lambda \psi\}_i \in \mathcal{W}(\mathfrak{g}, \Lambda_m, k)[\lambda]$.

To prove sesquilinearity, we need Lemma 4.20. The local bracket between $\partial\phi$ and ψ is:

$$(4.35) \quad \{\partial\phi(x), \psi(y)\}_i = \sum_{j \geq 0} \partial_x \frac{1}{j!} (\phi_{(j)} \psi)(y) \partial_y^j \delta(x - y) = \sum_{j \geq 0} -\frac{1}{j!} (\phi_{(j)} \psi)(y) \partial_y^{j+1} \delta(x - y)$$

Also, we have the bracket between ϕ and $\partial\psi$:

$$(4.36) \quad \begin{aligned} \{\phi(x), \partial\psi(y)\}_i &= \sum_{j \geq 0} \partial_y \frac{1}{j!} (\phi_{(j)} \psi)(y) \partial_y^j \delta(x - y) \\ &= \sum_{j \geq 0} \frac{1}{j!} \partial_y ((\phi_{(j)} \psi)(y)) \partial_y^j \delta(x - y) + \frac{1}{j!} (\phi_{(j)} \psi)(y) \partial_y^{j+1} \delta(x - y). \end{aligned}$$

Then the sesquilinearity

$$\{\partial\phi_\lambda \psi\}_i = -\lambda \{\phi_\lambda \psi\}_i, \quad \{\phi_\lambda \partial\psi\}_i = (\partial + \lambda) \{\phi_\lambda \psi\}_i$$

follow from the equation (4.32).

Leibniz rules, skew-symmetries and Jacobi identities of Poisson λ -brackets follow from those of local Poisson brackets. Hence $\mathcal{W}(\mathfrak{g}, \Lambda_m, k)$ endowed with the Poisson λ -bracket $\{\cdot_\lambda \cdot\}_i$ is a well-defined PVA. \square

Recall that

$$(4.37) \quad \{\phi(x), \psi(y)\}_1 = - \sum_{\substack{(i,j) \in \mathcal{I} \\ n \geq 0}} \sum_{\substack{(\alpha, \beta) \in \mathcal{I} \\ l \geq 0}} \frac{\partial \phi}{\partial u_i^{j(n)}}(x) \partial_x^n \frac{\partial \psi}{\partial u_\alpha^{\beta(l)}}(y) \partial_y^l [u_i^j, u_\alpha^{\beta+1}](y) \delta(x - y)$$

and

(4.38)

$$\begin{aligned} \{\phi(x), \psi(y)\}_2 &= \sum_{\substack{(i,0) \in \mathcal{I}, \\ n \geq 0}} \sum_{\substack{(\alpha,0) \in \mathcal{I}, \\ l \geq 0}} \frac{\partial \phi}{\partial u_i^{0(n)}}(x) \partial_x^n \frac{\partial \psi}{\partial u_\alpha^{0(l)}}(y) \partial_y^l (k(u_i^0, u_\alpha^0) \partial_y + [u_i^0, u_\alpha^0](y)) \delta(x-y) \\ &\quad - \sum_{\substack{(i,j) \in \mathcal{I}, \\ j > 0, n \geq 0}} \sum_{\substack{(\alpha,\beta) \in \mathcal{I}, \\ \beta > 0, l \geq 0}} \frac{\partial \phi}{\partial u_i^{j(n)}}(x) \partial_x^n \frac{\partial \psi}{\partial u_\alpha^{\beta(l)}}(y) \partial_y^l [u_i^j, u_\alpha^\beta](y) \delta(x-y). \end{aligned}$$

for $\phi, \psi \in \mathcal{V}_{[d,m]}$. These two local Poisson brackets are also well-defined on $\mathcal{W}_1(\mathfrak{g}, \Lambda, k)$. The brackets between basis elements in \mathcal{B} are

$$(4.39) \quad \{u_i^j(x), u_p^q(y)\}_1 = -z[u_i^j, u_p^q](y) \delta(x-y)$$

and

$$(4.40) \quad \{u_i^j(x), u_p^q(y)\}_2 = \begin{cases} k(u_i, u_p) \partial_y \delta(x-y) + [u_i, u_p](y) \delta(x-y) & \text{if } j = q = 0, \\ -[u_i^j, u_p^q](y) \delta(x-y) & \text{if } j \neq 0, q \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence the Poisson λ -brackets on $\mathcal{V}_{[d,m]}$ and $\mathcal{W}(\mathfrak{g}, \Lambda_m, k)$ are as follows:

Definition 4.22. The differential algebra $\mathcal{V}_{[d,m]}$ has two Poisson λ -brackets:

$$(4.41) \quad \{az_\lambda^i bz^j\}_1 = -[a, b] z^{i+j+1},$$

and

$$(4.42) \quad \{az_\lambda^i bz^j\}_2 = \begin{cases} k(a, b) \lambda + [a, b] & \text{if } i = j = 0, \\ 0 & \text{if } i = 0, j \neq 0, \text{ or } i \neq 0, j = 0, \\ -[az^i, bz^j] & \text{if } i \neq 0, j \neq 0, \end{cases}$$

where az^i and bz^j are basis elements in \mathcal{B} . Also, two brackets (4.41) and (4.42) induce well-defined Poisson λ -brackets on $\mathcal{W}(\mathfrak{g}, \Lambda_m, k) \subset \mathcal{V}_{[d,m]}$.

Furthermore, the following proposition shows that the two Poisson λ -brackets are compatible, which means that $\{\cdot, \cdot\}_\alpha := \{\cdot, \cdot\}_1 + \alpha \{\cdot, \cdot\}_2$ is a Poisson λ -bracket for any $\alpha \in \mathbb{C}$.

Proposition 4.23. *The Poisson λ -brackets $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ are compatible.*

Proof. Let us define the λ brackets with the parameter α by:

$$\{\cdot, \cdot\}_\alpha = \alpha \{\cdot, \cdot\}_1 + \{\cdot, \cdot\}_2.$$

The skewsymmetry, sesquilinearity and Leibniz rules of $\{\cdot, \cdot\}_\alpha$ are trivial by those of $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$. Thus the only thing to check is Jacobi identity. Due to the sesquilinearity and Leibniz rules, it suffices to show that:

$$(4.43) \quad \{az_\lambda^i \{bz_\lambda^j \mu cz_\lambda^k\}_\alpha\}_\alpha - \{bz_\lambda^j \mu \{az_\lambda^i \lambda cz_\lambda^k\}_\alpha\}_\alpha = \{\{az_\lambda^i \lambda bz_\lambda^j\}_\alpha \lambda + \mu cz_\lambda^k\}_\alpha,$$

when $a, b, c \in \mathfrak{g}$ and $i, j, k \in \mathbb{Z}_{\geq 0}$. If $i = j = k = 0$, then

$$\begin{aligned} \{az^i \lambda \{bz^j \mu cz^k\}_\alpha\}_\alpha &= (\alpha^2 z^2 - \alpha z + 1)[a, [b, c]] + k\lambda(a, [b, c]), \\ \{bz^j \mu \{az^i \lambda cz^k\}_\alpha\}_\alpha &= (\alpha^2 z^2 - \alpha z + 1)[b, [a, c]] - k\mu(b, [a, c]), \\ \{\{az^i \lambda bz^j\}_\alpha \lambda + \mu cz^k\}_\alpha &= (\alpha^2 z^2 - \alpha z + 1)[[a, b], c] + k(\lambda + \mu)([a, b], c), \end{aligned}$$

Similarly, one can check (4.43) in other cases. Hence Jacobi identity holds and the compatibility of $\{\cdot, \lambda \cdot\}_1$ and $\{\cdot, \lambda \cdot\}_2$ is proved. \square

5. GENERATING ELEMENTS OF A CLASSICAL AFFINE FRACTIONAL \mathcal{W} -ALGEBRA AND POISSON λ -BRACKETS BETWEEN THEM

5.1. Generating elements of classical affine fractional \mathcal{W} -algebras. Recall the universal Lax operator

$$(5.1) \quad \mathcal{L}_m = k\partial + Q_m + \Lambda_m \otimes 1 = k\partial + \sum_{(i,j) \in \mathcal{I}} \tilde{u}_i^{-j} \otimes u_i^j + \Lambda_m \otimes 1 \in \mathbb{C}\partial \ltimes \hat{\mathfrak{g}} \otimes \mathcal{V}_{[d,m]}.$$

We notice the map $\text{adf} : \mathfrak{g}(i) \rightarrow \mathfrak{g}(i-1)$, $i > 0$ is injective. Hence if $\mathfrak{b} = \bigoplus_{i \geq -\frac{1}{2}} \mathfrak{g}(i)$, then the adjoint map $\text{adf} : \mathfrak{n} \cdot z^{-m} \rightarrow \mathfrak{b} \cdot z^{-m}$ is also injective. Let us choose an $\text{ad}h$ -invariant complementary subspace V_m in $\mathfrak{b}z^{-m}$ such that

$$\mathfrak{b} \cdot z^{-m} = V_m \oplus [f, \mathfrak{n} \cdot z^{-m}]$$

and let us denote $\mathbb{C}_{\text{diff}}[V] = \mathbb{C}_{\text{diff}}[b_i | i \in J]$ when $\{b_i\}_{i \in J}$ is a basis of the vector space V .

Proposition 5.1. *Let $\overline{\mathbb{C}_{\text{diff}}[\hat{\mathfrak{g}}^m]}$ be the subspace $\mathbb{C}_{\text{diff}}[\hat{\mathfrak{g}}^m] + I$ of $\mathcal{V}_{[d,m]}$. Then there exist unique $S \in \mathfrak{n} \otimes \overline{\mathbb{C}_{\text{diff}}[\hat{\mathfrak{g}}^m]} \subset \mathfrak{n} \otimes \mathcal{V}_{[d,m]}$ and unique $Q_m^{\text{can}} \in (V_m \oplus \bigoplus_{-m < i \leq 0} \hat{\mathfrak{g}}^i) \otimes \mathcal{V}_{[d,m]}$ such that*

$$(5.2) \quad e^{\text{ad}S} \mathcal{L}_m = \mathcal{L}_m^{\text{can}} = k\partial + Q_m^{\text{can}} + \Lambda_m \otimes 1.$$

Proof. Equation (5.2) can be rewritten as

$$(5.3) \quad Q_m^{\text{can}} + [\Lambda_m \otimes 1, S] = Q_m + [S, k\partial + Q_m] + \frac{1}{2}[S, [S, \mathcal{L}_m]] + \frac{1}{6}[S, [S, [S, \mathcal{L}_m]]] + \dots$$

Since $\Lambda_m = -fz^{-m} - p^{-m-1}$ and $p \in \ker \text{ad}n$, equation (5.3) is same as

$$(5.4) \quad \begin{aligned} Q_m^{\text{can}} + [-fz^{-m} \otimes 1, S] &= Q_m + [S, k\partial + Q_m] \\ &+ \frac{1}{2}[S, [S, k\partial + Q_m - fz^{-m} \otimes 1]] + \frac{1}{6}[S, [S, [S, k\partial + Q_m - fz^{-m} \otimes 1]]] + \dots \end{aligned}$$

Let $S = \sum_{i>0} S_i$, $Q_m = \sum_{i>-(j+1)m-1, k \geq 0} Q_{i,k}$ and $Q_m^{\text{can}} = \sum_{i>-(j+1)m-1, k \geq 0} Q_{i,k}^{\text{can}}$, where $S_i \in \hat{\mathfrak{g}}_i^0 \otimes \mathcal{V}_{[d,m]}$, $Q_{i,k} \in \hat{\mathfrak{g}}_i^{-k} \otimes \mathcal{V}_{[d,m]}$ and $Q_{i,k}^{\text{can}} \in \hat{\mathfrak{g}}_i^{-k} \otimes \mathcal{V}_{[d,m]}$. By the injectivity of $\text{adf} : \mathfrak{n}z^{-m} \rightarrow \mathcal{B}z^{-m}$ and the equation

$$(5.5) \quad Q_{-(j+1)m-\frac{1}{2},m}^{\text{can}} + [fz^{-m} \otimes 1, S_{\frac{1}{2}}] = Q_{-(j+1)m-\frac{1}{2},m} \in \hat{\mathfrak{g}}_{-(d+1)m-\frac{1}{2}}^{-m} \otimes \overline{\mathbb{C}_{\text{diff}}[\hat{\mathfrak{g}}^m]}$$

which follows from equation (5.4), we can find $Q_{-(j+1)m-\frac{1}{2},m}^{\text{can}} \in (\hat{\mathfrak{g}}_{-(j+1)m-\frac{1}{2}}^{-m} \cap V_m) \otimes \overline{\mathbb{C}_{\text{diff}}[\hat{\mathfrak{g}}^m]}$ and $S_{\frac{1}{2}} \in \hat{\mathfrak{g}}_{\frac{1}{2}}^0 \otimes \overline{\mathbb{C}_{\text{diff}}[\hat{\mathfrak{g}}^m]}$ uniquely. Similarly, since $Q_{-(j+1)m,m}^{\text{can}}$ and S_1 are determined by

$Q_{-(j+1)m-\frac{1}{2},m}^{can}$, $S_{\frac{1}{2}}$, and Q_m , they should be in $\hat{\mathfrak{g}}_{-(j+1)m}^{-m} \otimes \overline{\mathbb{C}_{\text{diff}}[\hat{\mathfrak{g}}^m]}$ and $\hat{\mathfrak{g}}_1^0 \otimes \overline{\mathbb{C}_{\text{diff}}[\hat{\mathfrak{g}}^m]}$, respectively. By the induction on the gr_2 -grading, the whole S and the $(\hat{\mathfrak{g}}^{-m} \otimes \mathcal{V}_{[d,m]})$ -part of Q_m^{can} can be found uniquely. Once we find S , the $(\hat{\mathfrak{g}}^{-k} \otimes \mathcal{V}_{[d,m]})$ -part of Q_m^{can} , for any $k < m$, automatically come out from the equation $e^{\text{ad}S} \mathcal{L}_m = \mathcal{L}_m^{can}$. \square

Proposition 5.2. *Let $\{\tilde{w}_i^{-m} | i \in \mathcal{J}_m\}$ be a basis of V_m , where \tilde{w}_i is an eigenvector of $\text{ad } h$ and $\tilde{w}_i^{-m} := \tilde{w}_i z^{-m}$ and let $\{\tilde{u}_i^{-j} | (i,j) \in \mathcal{J}\} := \mathcal{B}^- \cap \bigoplus_{l < m} \hat{\mathfrak{g}}^{-l}$ be a basis of $\bigoplus_{-m < i \leq 0} \hat{\mathfrak{g}}^i$, where \mathcal{B}^- is defined in (4.10). Also, let us denote the dual elements of \tilde{w}_i^{-m} and \tilde{u}_i^{-j} by w_i^m and u_i^j , respectively. If we have*

$$Q_m^{can} = \sum_{i \in \mathcal{J}_m} \tilde{w}_i^{-m} \otimes \gamma_{w_i^m} + \sum_{(i,j) \in \mathcal{J}} \tilde{u}_i^{-j} \otimes \gamma_{u_i^j},$$

then $\mathcal{W}(\mathfrak{g}, \Lambda_m, k)$ is freely generated by $\gamma_{w_i^m}$ and $\gamma_{u_i^j}$ as a differential algebra. Moreover, the generating elements have the form

$$(5.6) \quad \gamma_{w_i^m} = w_i^m + A_{w_i^m}, \quad \gamma_{u_i^j} = u_i^j + B_{u_i^j},$$

where $A_{w_i^m} \in \mathbb{C}_{\text{diff}}[v \mid \text{gr}_1(v) = m, \text{gr}_2(v) > \text{gr}_2(w_i^m)]$ and $B_{u_i^j} \in \mathbb{C}_{\text{diff}}[v \mid \text{gr}_1(v) = m \text{ or } \text{gr}_1(u_i^j), \text{gr}_2(v) > \text{gr}_2(u_i^j)]$.

Proof. Equation (5.6) follows from equation (5.4) and the proof of Proposition 5.1. Hence the only thing to prove is that $\mathcal{W}(\mathfrak{g}, \Lambda_m, k) = \mathbb{C}_{\text{diff}}[\gamma_{w_l^m}(Q_m), \gamma_{u_l^j}(Q_m) \mid (i,j) \in \mathcal{J}, l \in \mathcal{J}_m]$. Suppose $\Phi \in \mathcal{V}_{[d,m]}$ is a gauge invariant functional. Then $\Phi(Q_m) = \Phi(Q_m^{can})$. Since $w_l^m(Q_m^{can}) = \gamma_{w_l^m}(Q_m)$, $u_l^j(Q_m^{can}) = \gamma_{u_l^j}(Q_m)$, we have

$$\begin{aligned} \Phi(Q_m) &= \Phi(Q_m^{can}) \in \mathbb{C}_{\text{diff}}[u_i^j(Q_m^{can}), w_l^m(Q_m^{can}) \mid (i,j) \in \mathcal{J}, l \in \mathcal{J}_m] \\ &= \mathbb{C}_{\text{diff}}[\gamma_{w_l^m}(Q_m), \gamma_{u_i^j}(Q_m) \mid (i,j) \in \mathcal{J}, l \in \mathcal{J}_m]. \end{aligned}$$

Hence we conclude that $\mathcal{W}(\mathfrak{g}, \Lambda_m, k) \subset \mathbb{C}_{\text{diff}}[\gamma_{w_l^m}(Q_m), \gamma_{u_i^j}(Q_m) \mid (i,j) \in \mathcal{J}, l \in \mathcal{J}_m]$.

On the other hand, Suppose that $Q_m \sim Q'_m$. Then $Q_m^{(can)} = Q'_m{}^{(can)}$. Since $\gamma_{u_i^j}(Q_m) = u_i^j(Q_m^{can}) = u_i^j(Q'_m{}^{can}) = \gamma_{u_i^j}(Q'_m)$ and $\gamma_{w_i^m}(Q_m) = w_i^m(Q_m^{can}) = w_i^m(Q'_m{}^{can}) = \gamma_{w_i^m}(Q'_m)$, we have $\mathbb{C}_{\text{diff}}[\gamma_{w_l^m}(Q_m), \gamma_{u_i^j}(Q_m) \mid (i,j) \in \mathcal{J}, l \in \mathcal{J}_m] \subset \mathcal{W}(\mathfrak{g}, \Lambda_m, k)$. \square

5.2. Examples.

Example 5.3. Let $\mathfrak{g} = \text{sl}_2$. The universal Lax operator for the m -th fractional \mathcal{W} -algebra $\mathcal{W}(\text{sl}_2, -fz^{-m} - ez^{-m-1}, 1)$ is

$$\begin{aligned} \mathcal{L}_m &= \partial + \sum_{j=0}^{m-1} (ez^{-j} \otimes fz^j + hz^{-j} \otimes xz^j + fz^{-j} \otimes ez^j) \\ &\quad + ez^{-m} \otimes fz^m + hz^{-m} \otimes xz^m + (fz^{-m} + ez^{-m-1}) \otimes -1. \end{aligned}$$

The gauge transformation of \mathcal{L}_m by $S = e \otimes A \in \mathfrak{n} \otimes \mathcal{V}_{[d,m]}$ is

$$e^{\text{ad}S} \mathcal{L}_m =: \mathcal{L}_m^{can} = (I \otimes 1 + e \otimes A)(\partial + Q_m + \Lambda_m \otimes 1)(I \otimes 1 - e \otimes A).$$

By Proposition 5.1, there exists a unique $q_m^{can} \in (\mathbb{C}fz^{-m} \oplus \bigoplus_{-m < i \leq 0} \hat{g}^i) \otimes \mathcal{V}_{[d,m]}$ which is gauge equivalent to q_m . We can write $q_m = \sum_{j=0}^m ez^{-j} \otimes fz^j(q_m) + hz^{-j} \otimes xz^j(q_m) + fz^{-j} \otimes ez^j(q_m)$ in a matrix form:

$$q_m = \sum_{j=0}^m \begin{bmatrix} xz^j(q_m) & fz^j(q_m) \\ ez^j(q_m) & -xz^j(q_m) \end{bmatrix} z^{-j}, \text{ where } ez^m(q_m) = 0.$$

Letting $e^{adS} \mathcal{L}_m = \partial + \tilde{q}_m + \Lambda \otimes 1$, the gauge equivalent element \tilde{q}_m to q_m can be written in a matrix form,

$$\begin{aligned} \tilde{q}_m = & \sum_{j=0}^{m-1} \begin{bmatrix} xz^j(q_m) + A(ez^j)(q_m) & fz^j(q_m) - 2A(xz^j)(q_m) - A^2(ez^j)(q_m) - \delta_{j,0}A' \\ ez^j(q_m) & -xz^j(q_m) - A(ez^j)(q_m) \end{bmatrix} z^{-j} \\ & + \begin{bmatrix} xz^m(q_m) - A(q_m) & fz^j(q_m) - 2A(xz^m)(q_m) + A^2(q_m) \\ 0 & -xz^j(q_m) + A(q_m) \end{bmatrix}. \end{aligned}$$

By substituting $A = xz^m$, we obtain

$$(5.7) \quad q_m^{can} + \Lambda_m = \sum_{j=0}^{m+1} \begin{bmatrix} \gamma_{xz^j}(q_m) & \gamma_{fz^j}(q_m) \\ \gamma_{ez^j}(q_m) & -\gamma_{xz^j}(q_m) \end{bmatrix} z^{-j},$$

where

$$(5.8) \quad \begin{aligned} \gamma_f &= f - 2(xz^m)x - (xz^m)^2e - \partial(xz^m); & \gamma_{fz^i} &= fz^i - 2(xz^m)(xz^i) - (xz^m)^2(ez^i); \\ \gamma_{xz^i} &= xz^j + (xz^m)(ez^j); & \gamma_{ez^j} &= ez^j; & \gamma_{fz^m} &= fz^m - (xz^m)^2; \\ \gamma_{xz^m} &= \gamma_{xz^{m+1}} = \gamma_{e^{m+1}} = 0; & \gamma_{ez^m} &= \gamma_{fz^{m+1}} = -1. \end{aligned}$$

Applying Proposition 5.2, we get a generating set $\{ \gamma_X \mid X = f, fz^i, xz^j, ez^j, fz^m, i = 1, \dots, m, j = 0, \dots, m-1 \}$ of $\mathcal{W}(\mathfrak{sl}_2, \Lambda_m = -fz^{-m} - ez^{-m-1}, 1)$.

The first λ -brackets between the elements in the generating set are as follows:

$$(5.9) \quad \begin{aligned} \{\gamma_f \lambda \gamma_f\}_1 &= -2\lambda, & \{\gamma_f \lambda \gamma_{fz^j}\}_1 &= -2\gamma_{xz^j}, \\ \{\gamma_f \lambda \gamma_{xz^i}\}_1 &= -\gamma_{fz^{i+1}} + \gamma_{ez^i}, & \{\gamma_f \lambda \gamma_{ez^i}\}_1 &= 2\gamma_{xz^{i+1}}, \end{aligned}$$

where $i \geq 0$ and $j \geq 1$. Otherwise,

$$(5.10) \quad \{\gamma_{g_1 z^i} \lambda \gamma_{g_2 z^j}\}_1 = -\gamma_{\{g_1, g_2\} z^{i+j+1}},$$

where $g_1, g_2 \in \mathfrak{sl}_2$ and $i, j \geq 0$.

The second λ -brackets are defined as follows:

$$(5.11) \quad \{\gamma_{g_1} \lambda \gamma_{g_2}\}_2 = \gamma_{\{g_1, g_2\}} + \lambda(g_1, g_2)$$

and

$$(5.12) \quad \begin{aligned} \{\gamma_{fz_\lambda} \gamma_f\}_2 &= -2\gamma_x - \lambda, & \{\gamma_{fz_\lambda} \gamma_x\}_2 &= \gamma_e, & \{\gamma_{fz_\lambda} \gamma_e\}_2 &= 0, & \{\gamma_{fz_\lambda} \gamma_{fz}\}_2 &= 0, \\ \{\gamma_{fz_\lambda} \gamma_{fz^j}\}_2 &= -2\gamma_{xz^j}, & \{\gamma_{fz_\lambda} \gamma_{xz^i}\}_2 &= -\gamma_{fz^{1+i}} + \gamma_{ez^i}, & \{\gamma_{fz_\lambda} \gamma_{ez^i}\}_2 &= 2\gamma_{xz^{1+i}}, \end{aligned}$$

where $g_1, g_2 \in \mathfrak{sl}_2$, $j \geq 2$ and $i, l \geq 1$. Otherwise,

$$\{\gamma_{g_1 z^i} \lambda \gamma_{g_2 z^l}\}_2 = -\gamma_{\{g_1 z^i, g_2 z^l\}}.$$

Example 5.4. Let $\mathfrak{g} = \mathfrak{sl}_n$ and $\Lambda_m = -e_{n1}z^{-m} - e_{1n}z^{-m-1}$. Consider the dual bases

$$\mathcal{B} = \{ E_{ij}^k, E_{ll}^k \mid i \neq j, l = 1, \dots, n-1, k = 0, \dots, m \} \setminus \{ e_{1n}z^m \}$$

and

$$\mathcal{B}^- = \{ \tilde{E}_{ij}^{-k}, \tilde{E}_{ll}z^{-k} \mid i \neq j, l = 1, \dots, n-1, k = 0, \dots, m \} \setminus \{ e_{n1}z^{-m} \}$$

of $\hat{\mathfrak{g}}_{\leq -2m+1}^{\geq 0}$ and $\hat{\mathfrak{g}}_{> -2m-1}^{\leq 0}$, where $E_{ij}^k := e_{ji}z^k$, $E_{ll}^k := \frac{n-1}{n}e_{ll}z^k - \frac{1}{n}e_{nn}z^k$, $\tilde{E}_{ij}^{-k} := e_{ij}z^{-k}$ and $\tilde{E}_{ll}z^{-k} := (e_{ll} - e_{nn})z^{-k}$. The universal Lax operator $\mathcal{L}_m = \partial + \sum_{E_{ij}^k \in \mathcal{B}} \tilde{E}_{ij}^{-k} \otimes E_{ij}^k + \Lambda_m \otimes 1 \in \mathbb{C}\partial \ltimes \hat{\mathfrak{sl}}_n \otimes \mathcal{V}_{[d,m]}$ can be written in the matrix form as below:

$$(5.13) \quad \mathcal{L}_m = \partial + \sum_{k=0}^{m-1} z^{-k} \left(E_{i,j}^k \right)_{i,j=1,\dots,n} - z^{-m}(e_{n1} - z^{-1}e_{1n}), \quad \text{where } E_{nn}^k = -\sum_{i=1}^{n-1} E_{ii}^k.$$

Then an element in the associated Lie group to the Lie algebra $\mathfrak{n} \otimes \mathcal{V}_{[d,m]}$ is an upper diagonal matrix.
Suppose

$$(5.14) \quad S = \begin{bmatrix} 1 & s_{12} & \cdots & s_{1n} \\ 0 & 1 & 0 & s_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix} \in \mathfrak{n} \otimes \mathcal{V}_{[d,m]}$$

and $q_m \sim_S \bar{q}_m$, i.e. $\bar{q}_m = S(q + \Lambda_m)S^{-1} - \Lambda_m + S\partial(S^{-1})$. In the matrix form, $\bar{q}_m = \sum_{k=0}^m z^{-k}(a_{ij}^k)_{i,j=1}^n$, where

$$\begin{aligned}
(5.15) \quad & a_{n1}^q = E_{n1}^q, \quad a_{i1}^r = E_{i1}^r + s_{in}E_{n1}^r, \quad a_{11}^r = E_{11}^r + \sum_{k=2}^n s_{ik}E_{k1}^r, \quad a_{nj}^r = E_{nj}^r - E_{n1}^r s_{1j}, \\
& a_{ij}^r = E_{ij}^r + s_{in}E_{nj}^r - (E_{i1}^r + s_{in}E_{n1}^r)s_{1j}, \quad a_{1j}^p = E_{1j}^p - \sum_{k=2}^n s_{1k}E_{kj}^p - (E_{11}^p + \sum_{k=2}^n s_{1k}E_{k1}^p)s_{1j}, \\
& a_{1j}^0 = E_{1j}^0 - \sum_{k=2}^n s_{1k}E_{kj}^0 - (E_{11}^0 + \sum_{k=2}^n s_{1k}E_{k1}^0)s_{1j} - ks'_{1j}, \\
& a_{in}^p = E_{in}^p + s_{in}E_{nn}^p + (E_{i1}^p + \sum_{k=2}^n s_{ik}E_{k1}^p)t_{1n} - \sum_{j=2}^{n-1} (E_{ij}^p + \sum_{k=2}^n s_{ik}E_{kj}^p)s_{jn}, \\
& a_{in}^0 = E_{in}^p + s_{in}E_{nn}^p + (E_{i1}^p + \sum_{k=2}^n s_{ik}E_{k1}^p)t_{1n} - \sum_{j=2}^{n-1} (E_{ij}^p + \sum_{k=2}^n s_{ik}E_{kj}^p)s_{jn} - ks'_{in}, \\
& a_{1n}^p = E_{1n}^p + \sum_{k=2}^n s_{1k}E_{kn}^p + (E_{11}^p + \sum_{k=2}^n s_{1k}E_{k1}^p)t_{1n} - \sum_{j=2}^{n-1} (E_{1j}^p + \sum_{k=2}^n s_{1k}E_{kj}^p)s_{jn}, \\
& a_{1n}^0 = E_{1n}^p + \sum_{k=2}^n s_{1k}E_{kn}^p + (E_{11}^p + \sum_{k=2}^n s_{1k}E_{k1}^p)t_{1n} - \sum_{j=2}^{n-1} (E_{1j}^p + \sum_{k=2}^n s_{1k}E_{kj}^p)s_{jn} + kv_{1n}, \\
& a_{nn}^r = -\sum_{i=1}^{n-1} a_{ii}^r, \quad E_{n1}^m = -1.
\end{aligned}$$

Here $t_{1n} = -s_{1n} + s_{12}s_{2n} + \cdots + s_{1n-1}s_{n-1n}$ and $v_{1n} = -\partial s_{1n} + \partial(s_{12})s_{2n} + \cdots + \partial(s_{1n-1})s_{n-1n}$ and $p = 1, \dots, m$, $q = 0, \dots, m-1$, $r = 0, \dots, m$, $i, j = 2, \dots, n-1$. If the entries of S in (5.14) are picked as follows:

$$(5.16) \quad s_{in} = \begin{cases} -\frac{E_{i1}^m}{E_{n1}^m} & \text{if } E_{n1}^m \neq 0 \\ 0 & \text{otherwise} \end{cases}, \quad s_{1j} = \begin{cases} \frac{E_{nj}^m}{E_{n1}^m} & \text{if } E_{n1}^m \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Then we get the generators $\gamma_{ij}^k(q_m) := E_{ij}^k(\bar{q}_m)$ of the differential algebra $\mathcal{W}(\mathfrak{sl}_n, \Lambda_m, 1)$.

5.3. More results on generating elements of a classical affine fractional \mathcal{W} -algebra associated to a minimal nilpotent. Suppose f is a minimal nilpotent and $m \in \mathbb{Z}_{>0}$. Under this assumption, we can describe generating elements of $\mathcal{W}(\mathfrak{g}, \Lambda_m, k)$ and λ -brackets between them. The following lemma is useful to find a generating set of $\mathcal{W}(\mathfrak{g}, \Lambda_m, k)$.

Lemma 5.5. *Let $X := \{ w_l^m \mid l \in \mathcal{J}_m \}$ and $Y := \{ u_i^j \mid (i, j) \in \mathcal{J} \}$ be defined as in Proposition 5.2 and let $\psi : \mathfrak{g}_f \cdot z^m \oplus \bigoplus_{i=0}^{m-1} \mathfrak{g} \cdot z^i \rightarrow \mathcal{W}(\mathfrak{g}, \Lambda_m, k)$ be a linear map such that $\psi_v := \psi(v) = v + C_v$, where $v \in X \cup Y$ and $C_v \in \mathbb{C}[u, u', u'', \dots \mid u \in X \cup Y, \text{gr}_2(u) > \text{gr}_2(v)]$. Then $\Psi := \{ \psi_v \mid v \in X \cup Y \}$ freely generates $\mathcal{W}(\mathfrak{g}, \Lambda_m, k)$.*

Proof. Since we know that the set $\{ \gamma_v \mid v \in X \cup Y \}$ defined in (5.6) freely generates $\mathcal{W}(\mathfrak{g}, \Lambda_m, k)$, it suffices to show that any γ_v is in the differential algebra generated by Ψ . We know that $\gamma_v - \psi_v$ is an element in $\mathcal{W}(\mathfrak{g}, \Lambda_m, k)$. Hence, by Proposition 5.2 and the assumption on C_v , the element $\gamma_v - \psi_v \in \mathbb{C}_{\text{diff}}[\gamma_u \mid u \in X \cup Y, \text{gr}_2(u) > \text{gr}_2(v)]$. By the induction on gr_2 -grading, we have $\gamma_v - \psi_v \in \Psi$ so that $\gamma_v \in \Psi$. \square

In order to state and prove the main theorems in this section (see Theorem 5.6 and 5.8), take two bases $\{z_i\}_{i=1}^{2s}$ and $\{z_i^*\}_{i=1}^{2s}$ of $\mathfrak{g}(\frac{1}{2})$ such that $[z_i, z_j^*] = \delta_{ij}e$ and let $z_{2s+1} := x$ and $z_{2s+1}^* := e$. Also, we let $\{\cdot, \cdot\}$ be the Poisson bracket on $\mathcal{V}_{[d,m]}$ induced from the Lie bracket on $\hat{\mathfrak{g}}$.

Theorem 5.6. *Let $\tilde{\eta}_m : \hat{\mathfrak{g}}_{[d,m]}^+ \rightarrow \mathcal{V}_{[d,m]}$ be a linear map such that*

$$(5.17) \quad \tilde{\eta}_m(a) = \sum_{l=0}^4 \sum_{i_1, \dots, i_l=1}^{2s+1} \frac{1}{l!} (z_{i_1} z^m) \cdots (z_{i_l} z^m) \{a, z_{i_1}^*, \dots, z_{i_l}^*\},$$

where $\{a_1, a_2, \dots, a_t\}$ is the $(t-1)$ -brackets $\{\{\dots\{a_1, a_2\}, \dots, a_{t-1}\}, a_t\}$, and let $\eta_m : \hat{\mathfrak{g}}_{[d,m]}^+ \rightarrow \mathcal{V}_{[d,m]}$ be a linear map such that

$$(5.18) \quad \begin{aligned} \eta_m(a) &= \tilde{\eta}_m(a) \quad \text{if } a \in \hat{\mathfrak{g}}_{\geq 0}^0 \oplus \left(\bigoplus_{t=1}^{m-1} \hat{\mathfrak{g}}^t \right) \oplus \mathfrak{g}_f z^m, \\ \eta_m(b) &= \tilde{\eta}_m(b) - \sum_{i=1}^{2s} (z_i^*, w) k \partial(z_i z^m) \quad \text{if } b \in \hat{\mathfrak{g}}_{-\frac{1}{2}}^0, \\ \eta_m(f) &= \tilde{\eta}_m(f) - k \partial(x z^m) - \frac{k}{2} \sum_{i=1}^{2s} \partial(z_i^* z^m) (z_i z^m). \end{aligned}$$

Then

$$\mathbb{C}_{\text{diff}} \left[\eta_m \left(\bigoplus_{i=0}^{m-1} \mathfrak{g} z^i \oplus \mathfrak{g}_f z^m \right) \right] = \mathcal{W}(\mathfrak{g}, \Lambda_m, k).$$

Equivalently, Theorem 5.6 can be written as follows.

Theorem 5.7. *Suppose that $u \in \mathfrak{g}(\frac{1}{2}), v \in \mathfrak{g}(0), w \in \mathfrak{g}(-\frac{1}{2}), t = 0, \dots, m-1$, and $v_f \in \mathfrak{g}_f(0), w_f \in \mathfrak{g}_f(-\frac{1}{2})$. Then the following elements freely generate $\mathcal{W}(\mathfrak{g}, \Lambda_m, k)$ as a*

differential algebra :

$$\begin{aligned}
\eta_m(ez^t) &= ez^t; \\
\eta_m(uz^t) &= uz^t + \sum_{i=1}^{2s} (z_i z^m) \{uz^t, z_i^*\}; \\
\eta_m(vz^t) &= vz^t + \sum_{i=1}^{2s} (z_i z^m) \{vz^t, z_i^*\} + (xz^m) \{v, e\} + \frac{1}{2} \sum_{i,j=1}^{2s} (z_j z^m) (z_i z^m) \{vz^t, z_i^*, z_j^*\}; \\
\eta_m(wz^t) &= wz^t + \sum_{i=1}^{2s} (z_i z^m) \{wz^t, z_i^*\} + (xz^m) \{wz^t, e\} + \frac{1}{2} \sum_{i,j=1}^{2s} (z_j z^m) (z_i z^m) \{wz^t, z_i^*, z_j^*\} \\
&\quad + \sum_{i=1}^{2s} (xz^m) (z_i z^m) \{wz^t, e, z_i^*\} + \frac{1}{6} \sum_{i,j,k=1}^{2s} (z_i z^m) (z_j z^m) (z_k z^m) \{w^t, z_i^*, z_j^*, z_k^*\} \\
&\quad - \delta_{t,0} \left(\sum_{i=1}^{2s} (z_i^*, w) k \partial (z_i z^m) \right); \\
\eta_m(fz^t) &= fz^t + \sum_{i=1}^{2s} (z_i z^m) \{fz^t, z_i^*\} - 2(xz^m)(xz^t) + \frac{1}{2} \sum_{i,j=1}^{2s} (z_j z^m) (z_i z^m) \{fz^t, z_i^*, z_j^*\} \\
&\quad - 2 \sum_{i=1}^{2s} (xz^m) (z_i z^m) \{xz^t, z_i^*\} - (xz^m)^2 (ez^t) + \frac{1}{6} \sum_{i,j,k=1}^{2s} (z_i z^m) (z_j z^m) (z_k z^m) \{fz^t, z_i^*, z_j^*, z_k^*\} \\
&\quad - \sum_{i,j=1}^{2s} (xz^m) (z_i z^m) (z_j z^m) \{xz^t, z_i^*, z_j^*\} \\
&\quad + \frac{1}{24} \sum_{i,j,k,l=1}^{2s} (z_i z^m) (z_j z^m) (z_k z^m) (z_l z^m) \{fz^t, z_i^*, z_j^*, z_k^*, z_l^*\} \\
&\quad - \delta_{t,0} \left(k \partial (xz^m) + \frac{k}{2} \sum_{i=1}^{2s} \partial (z_i^* z^m) (z_i z^m) \right);
\end{aligned}$$

and

$$\begin{aligned}
\eta_m(v_f z^m) &= v_f z^m + \frac{1}{2} \sum_{i=1}^{2s} (z_i z^m) \{v_f z^m, z_i^*\}; \\
\eta_m(w_f z^m) &= w_f z^m + \sum_{i=1}^{2s} (z_i z^m) \{w_f z^m, z_i^*\} + \frac{1}{3} \sum_{i,j=1}^{2s} (z_j z^m) (z_i z^m) \{w_f z^m, z_i^*, z_j^*\}; \\
\eta_m(f z^m) &= f z^m + \sum_{i=1}^{2s} (z_i z^m) \{f z^m, z_i^*\} - (x z^m)^2 + \frac{1}{2} \sum_{i,j=1}^{2s} (z_j z^m) (z_i z^m) \{f z^m, z_i^*, z_j^*\} \\
&\quad + \frac{1}{6} \sum_{i,j,k=1}^{2s} (z_i z^m) (z_j z^m) (z_k z^m) \{f z^m, z_i^*, z_j^*, z_k^*\} \\
&\quad + \frac{1}{24} \sum_{i,j,k,l=1}^{2s} (z_i z^m) (z_j z^m) (z_k z^m) (z_l z^m) \{f z^m, z_i^*, z_j^*, z_k^*, z_l^*\}.
\end{aligned}$$

Proof. By Lemma 5.5, the only thing to check is that the $\text{ad}_\lambda \mathfrak{n}$ -action trivially acts on $\eta_m \left(\bigoplus_{i=0}^{m-1} \mathfrak{g} z^i \oplus \mathfrak{g} f z^m \right)$. For example, in the case when $w \in \mathfrak{g} \left(\frac{1}{2} \right)$ and $t > 0$, we have

$$\begin{aligned}
\{z_l^* \lambda \sum_{i=1}^{2s} (z_i z^m) \{w z^t, z_i^*\}\} &= \{w z^t, z_l^*\} + \sum_{i=1}^{2s} (z_i z^m) \{z_l^*, \{w z^t, z_i^*\}\}; \\
\{z_l^* \lambda \sum_{i,j=1}^{2s} 2s (z_i z^m) (z_j z^m) \{w z^t, z_i^*, z_j^*\}\} &= 2 \sum_{i=1}^{2s} (z_i z^m) \{w z^t, z_i^*, z_l^*\} \\
&\quad - \sum_{i,j=1}^{2s} (z_i z^m) (z_j z^m) \{w z^t, z_i^*, z_j^*, z_k^*\} - \{z_l^*, \{v, e\} (2x z^m)\} \\
&\quad + \{z_l^*, \sum_{i=1}^{2s} \{z_i^*, \{v, e\}\} (z_i z^m) (2x z^m)\} + \sum_{i=1}^{2s} \{z_i^*, \{v, e\}\} (z_i z^m) (z_l^* z^m);
\end{aligned}$$

and

$$\begin{aligned}
&\{z_l^* \lambda \sum_{i,j,k=1}^{2s} (z_i z^m) (z_j z^m) (z_k z^m) \{v, z_i^*, z_j^*, z_k^*\}\} \\
&= 3 \sum_{i,j=1}^{2s} (z_i z^m) (z_j z^m) \{v, z_i^*, z_j^*, z_l^*\} + 3 \sum_{i=1}^{2s} \{v, z_i^*, e\} (z_i z^m) (z_l^* z^m).
\end{aligned}$$

Hence $\{z_l^* \lambda \eta_m(w z^t)\} = 0$. The most complicated case to check is $\{z_l^* \lambda \eta_m(f z_k)\} = 0$. However, since $\{\eta_m(v z^i) \lambda \eta_m(w z^j)\} = \eta_m(\{v, w\}, e) f z^{i+j}$, by Jacobi identity, we have $\{z_l^* \lambda \eta_m(f z_k)\} = 0$. \square

Theorem 5.8. Assume that $m > 0$. The first λ -brackets between the generating elements are

$$\begin{aligned}\{\eta_m(f) \lambda \eta_m(f)\}_1 &= -2k\lambda; \\ \{\eta_m(f) \lambda \eta_m(uz^i)\}_1 &= -\eta_m([f, uz^i]z) + \eta_m([uz^i, e]), \quad \text{if } uz^i \neq f; \\ \{\eta_m(az^i) \lambda \eta_m(bz^j)\}_1 &= -\eta_m([a, b]z^{i+j+1}), \quad \text{otherwise};\end{aligned}$$

and the second λ -brackets are

$$\begin{aligned}\{\eta_m(a) \lambda \eta_m(b)\}_2 &= \eta_m([a, b]) + k\lambda(a, b); \\ \{\eta_m(fz) \lambda \eta_m(f)\}_2 &= -2\eta_m(x) - k\lambda; \\ \{\eta_m(fz) \lambda \eta_m(u)\}_2 &= \eta_m([u, e]); \\ \{\eta_m(fz) \lambda \eta_m(az^i)\}_2 &= -\eta_m([fz, az^i]) + \eta_m([az^i, e]); \\ \{\eta_m(az^i) \lambda \eta_m(bz^j)\}_2 &= -\eta_m([a, b]z^{i+j});\end{aligned}$$

for $a, b \in \mathfrak{g}$, $u \in \bigoplus_{i=-\frac{1}{2}}^1 \mathfrak{g}(i)$, and $i, j > 0$.

We need the following remark and the lemma to show Theorem 5.8.

Remark 5.9. Recall that $\{\cdot \lambda \cdot\}_1$ and $\{\cdot \lambda \cdot\}_2$ are the λ -brackets on $\mathcal{V}_{[d, m]}$. We have

- (i) $\{\mathbf{nz}^m \lambda \mathcal{V}_{[d, m]}\}_1 = \{\mathbf{nz}^m \lambda \mathcal{V}_{[d, m]}\}_2 = 0$,
- (ii) if $a \in \bigoplus_{i>-1} \mathfrak{g}(i) \oplus \bigoplus_{t>0} \hat{\mathfrak{g}}^t$ then $\{xz_\lambda^m a\}_1 = 0$ and $\{xz_\lambda^m f\}_1 = fz^{m+1} = -1$,
- (iii) if $a \in \mathfrak{g} \oplus \bigoplus_{i>-1} \mathfrak{g}(i)z \oplus \bigoplus_{t>1} \hat{\mathfrak{g}}^t$ then $\{xz_\lambda^m a\}_1 = 0$ and $\{xz_\lambda^m fz\}_2 = fz^{m+1} = -1$.

In $m = 0$ case, (i) and (iii) are not true since $\{z_i \lambda z_j^*\}_2 = -1$ and $\{x_\lambda z_i\}_2 = \frac{1}{2}z_i$.

Lemma 5.10. Let $v, w \in \mathcal{V}_{[d, m]}$. Then for any $k \geq 0$, the following equation holds:

$$\begin{aligned}(5.19) \quad & \sum_{i_1, \dots, i_r=1}^{2s+1} \frac{1}{r!} \{\{v, w\}, z_{i_1}^*, z_{i_2}^*, \dots, z_{i_r}^*\} \\ &= \sum_{l=0}^r \frac{1}{l!} \frac{1}{(r-l)!} \sum_{i_1, \dots, i_r=1}^{2s+1} \{v, z_{i_1}^*, z_{i_2}^*, \dots, z_{i_l}^*\} \{w, z_{i_{l+1}}^*, z_{i_{l+2}}^*, \dots, z_{i_r}^*\}.\end{aligned}$$

Proof. If $r = 0$, equation (5.19) holds obviously. Suppose that we have (5.19) when $r = n$. Then by Jacobi identity, we have

$$\begin{aligned}
(5.20) \quad & \sum_{i_1, \dots, i_{n+1}=1}^{2s+1} \frac{1}{(n+1)!} \{ \{ \{ v, w \}, z_{i_1}^* \}, z_{i_2}^*, \dots, z_{i_{n+1}}^* \} \\
&= \frac{1}{n+1} \sum_{l=0}^n \frac{1}{l!} \frac{1}{(n-l)!} \sum_{i_1, \dots, i_{n+1}=1}^{2s+1} \{ v, z_{i_1}^*, z_{i_2}^*, \dots, z_{i_l}^*, z_{i_{n+1}}^* \} \{ w, z_{i_{l+1}}^*, z_{i_{l+2}}^*, \dots, z_{i_n}^* \} \\
&+ \frac{1}{n+1} \sum_{l=0}^n \frac{1}{l!} \frac{1}{(n-l)!} \sum_{i_1, \dots, i_{n+1}=1}^{2s+1} \{ v, z_{i_1}^*, z_{i_2}^*, \dots, z_{i_l}^* \} \{ w, z_{i_{l+1}}^*, z_{i_{l+2}}^*, \dots, z_{i_n}^*, z_{i_{n+1}}^* \} \\
&= \frac{1}{n+1} \sum_{l=0}^n \frac{(l+1) + (n-l)}{(l+1)!(n-l)!} \sum_{i_1, \dots, i_{n+1}=1}^{2s+1} \{ v, z_{i_1}^*, z_{i_2}^*, \dots, z_{i_l}^*, z_{i_{n+1}}^* \} \{ w, z_{i_{l+1}}^*, z_{i_{l+2}}^*, \dots, z_{i_n}^* \} \\
&+ \frac{1}{(n+1)!} \sum_{i_1, \dots, i_{n+1}=1}^{2s+1} \{ v, \{ w, z_1^*, \dots, z_{i_{n+1}}^* \} \} \\
&= \sum_{l=0}^{n+1} \frac{1}{l!} \frac{1}{((n+1)-l)!} \sum_{i_1, \dots, i_{n+1}=1}^{2s+1} \{ v, z_{i_1}^*, z_{i_2}^*, \dots, z_{i_l}^* \} \{ w, z_{i_{l+1}}^*, z_{i_{l+2}}^*, \dots, z_{i_{n+1}}^* \}.
\end{aligned}$$

Hence (5.19) holds for any $r \geq 0$. □

Proof of Theorem 5.8 :

If $a, b \in \bigoplus_{i>-1} \mathfrak{g}(i) \oplus (\bigoplus_{t>0} \hat{\mathfrak{g}}^t)$, by Remark 5.9 and Lemma 5.10, we have

$$\begin{aligned}
(5.21) \quad & \{ \eta_m(a)_\lambda \eta_m(b) \}_1 = \{ \tilde{\eta}_m(a)_\lambda \tilde{\eta}_m(b) \}_1 \\
&= \sum_{r=0}^4 \sum_{i_1, \dots, i_r=1}^{2s+1} \frac{1}{r!} ((z_{i_1} z^m) \cdots (z_{i_r} z^m)) (-\{ \{ a, b \}, z_{i_1}^*, z_{i_2}^*, \dots, z_{i_r}^* \} z) \\
&= -\tilde{\eta}_m([a, b]z) = -\eta_m([a, b]z).
\end{aligned}$$

Also, we have

$$\begin{aligned}
(5.22) \quad & \{ \eta_m(f)_\lambda \eta_m(f) \}_1 = -\{ \tilde{\eta}_m(f), \tilde{\eta}_m(f) \} z + \{ f_\lambda (-k \partial(xz^m)) \}_1 + \{ (-k \partial(xz^m))_\lambda f \}_1 \\
&= \{ f_\lambda (-k \partial(xz^m)) \}_1 + \{ (-k \partial(xz^m))_\lambda f \}_1 = -2k\lambda,
\end{aligned}$$

and, if $b \in \bigoplus_{i>-1} \mathfrak{g}(i) \oplus (\bigoplus_{t>0} \hat{\mathfrak{g}}^t)$, we obtain
(5.23)

$$\begin{aligned} & \{\eta_m(f)_\lambda \eta_m(b)\}_1 \\ &= -\tilde{\eta}_m([f, b]z) + \sum_{p=0}^2 \sum_{i_i, \dots, i_p=1}^{2s} \frac{1}{p!} (z_{j_1} z^m) \cdots (z_{j_p} z^m) \{\{b, e\}, z_{i_1}^*, \dots, z_{i_p}^*\} + \{b, e, e\}(xz^m) \\ &= -\tilde{\eta}_m([f, b]z) + \tilde{\eta}_m([b, e]) = -\eta_m([f, b]z) + \eta_m([b, e]). \end{aligned}$$

Next, let us compute the second λ -brackets. If $a, b \in \mathfrak{g}$, we have

$$(5.24) \quad \{\eta_m(a)_\lambda \eta_m(b)\}_2 = \{\tilde{\eta}_m(a)_\lambda \tilde{\eta}_m(b)\}_2$$

by Remark 5.9, and

$$\begin{aligned} & \{\tilde{\eta}_m(a)_\lambda \tilde{\eta}_m(b)\}_2 \\ &= \tilde{\eta}_m([a, b]) + \sum_{r=0}^4 \sum_{l=0}^r \sum_{i_1, \dots, i_r=1}^{2s+1} ((z_{i_{l+1}} z^m) \cdots (z_{i_r} z^m)) \\ (5.25) \quad & \cdot \left(k(\lambda + \partial)(\{a, z_{i_1}^*, z_{i_2}^*, \dots, z_{i_l}^*\}, \{b, z_{i_{l+1}}^*, z_{i_{l+2}}^*, \dots, z_{i_r}^*\}) ((z_{i_1} z^m) \cdots (z_{i_l} z^m)) \right) \\ &= \tilde{\eta}_m([a, b]) + k(\lambda + \partial)(a, b) + X_0 + \lambda X_1 = \eta_m([a, b]) + k\lambda(a, b) + Y_0 + \lambda Y_1, \end{aligned}$$

where $X_0, X_1, Y_0, Y_1 \in \mathbb{C}_{\text{diff}}[(\mathfrak{g}(\frac{1}{2}) \oplus \mathfrak{g}(1)) \cdot z^m]$ with zero constant term. Since $\{\eta_m(a)_\lambda \eta_m(b)\}_2$ and $\eta_m([a, b]) + k\lambda(a, b)$ are in $\mathcal{W}(\mathfrak{g}, \Lambda_m, k)[\lambda]$, the element Y_0 and Y_1 should be in $\mathcal{W}(\mathfrak{g}, \Lambda_m, k)$. However, we know that $(\mathbb{C}_{\text{diff}}[(\mathfrak{g}(\frac{1}{2}) \oplus \mathfrak{g}(1)) \cdot z^m] \cap \mathcal{W}(\mathfrak{g}, \Lambda_m, k)) = \mathbb{C}$. Hence $Y_0 = Y_1 = 0$.

If $a, b \in \bigoplus_{i>-1} \mathfrak{g}(i)z \oplus \bigoplus \hat{\mathfrak{g}}^{\geq 2}$, then

$$\begin{aligned} & \{\eta_m(a)_\lambda \eta_m(b)\}_2 = \{\tilde{\eta}_m(a)_\lambda \tilde{\eta}_m(b)\}_2 \\ (5.26) \quad &= \sum_{r=0}^4 \sum_{l=0}^r \sum_{i_1, \dots, i_r=1}^{2s+1} \frac{1}{l!} \frac{1}{(r-l)!} ((z_{i_1} z^m) \cdots (z_{i_l} z^m)) ((z_{i_{l+1}} z^m) \cdots (z_{i_r} z^m)) \\ & \quad \cdot (-\{ \{a, z_{i_1}^*, z_{i_2}^*, \dots, z_{i_l}^*\}, \{b, z_{i_{l+1}}^*, \dots, z_{i_r}^*\} \}) \\ &= -\tilde{\eta}_m([a, b]) = -\eta_m([a, b]). \end{aligned}$$

The second λ -bracket between $\eta_m(fz)$ and $\eta_m(f)$ is
(5.27)

$$\begin{aligned}
\{\eta_m(fz)_\lambda \eta_m(f)\}_2 &= \sum_{p=0}^2 \sum_{i_i, \dots, i_p=1}^{2s} \frac{1}{p!} (z_{j_1} z^m) \cdots (z_{j_p} z^m) \{\{f, e\}, z_{i_1}^*, \dots, z_{i_p}^*\} \{fz_\lambda (xz^m)\}_2 \\
&\quad + \{f, e, e\} \left\{ fz_\lambda \frac{1}{2} (xz^m)^2 \right\}_2 - k \{fz_\lambda \partial (xz^m)\}_2 \\
&= \sum_{p=0}^2 \sum_{i_i, \dots, i_p=1}^{2s} \frac{1}{p!} (z_{j_1} z^m) \cdots (z_{j_p} z^m) \{-2x, z_{i_1}^*, \dots, z_{i_p}^*\} + \{-2x, e\} (xz^m) - k\lambda \\
&= -2\eta_m(x) - k\lambda.
\end{aligned}$$

If $u \in \bigoplus_{i \geq -\frac{1}{2}} \mathfrak{g}(i)$, then
(5.28)

$$\begin{aligned}
\{\eta_m(fz)_\lambda \eta_m(u)\}_2 &= \sum_{p=0}^2 \sum_{i_i, \dots, i_p=1}^{2s} \frac{1}{p!} (z_{j_1} z^m) \cdots (z_{j_p} z^m) \{\{u, e\}, z_{i_1}^*, \dots, z_{i_p}^*\} \{fz_\lambda (xz^m)\}_2 \\
&= \sum_{p=0}^2 \sum_{i_i, \dots, i_p=1}^{2s} \frac{1}{p!} (z_{j_1} z^m) \cdots (z_{j_p} z^m) \{\{u, e\}, z_{i_1}^*, \dots, z_{i_p}^*\} = \eta_m([u, e]).
\end{aligned}$$

If $b \in \bigoplus_{i \geq 0} \hat{\mathfrak{g}}^i$, then
(5.29)

$$\begin{aligned}
\{\eta_m(fz)_\lambda \eta_m(b)\}_2 &= \sum_{r=0}^4 \sum_{l=0}^r \sum_{i_1, \dots, i_r=1}^{2s+1} \frac{1}{l!} \frac{1}{(r-l)!} ((z_{i_1} z^m) \cdots (z_{i_l} z^m)) ((z_{i_{l+1}} z^m) \cdots (z_{i_r} z^m)) \\
&\quad \cdot (-[\{fz, z_{i_1}^*, z_{i_2}^*, \dots, z_{i_l}^*\}, \{b, z_{i_{l+1}}^*, z_{i_{l+2}}^*, \dots, z_{i_r}^*\}]) \\
&\quad + \sum_{p=0}^2 \sum_{i_i, \dots, i_p=1}^{2s} \frac{1}{p!} (z_{j_1} z^m) \cdots (z_{j_p} z^m) \{\{b, e\}, z_{i_1}^*, \dots, z_{i_p}^*\} \{fz_\lambda (xz^m)\}_2 \\
&\quad + \{b, e, e\} \left\{ fz_\lambda \frac{1}{2} (xz^m)^2 \right\}_2 \\
&= -\eta_m([fz, b]) + \eta_m([b, e]).
\end{aligned}$$

So we proved Theorem 5.8.

6. INTEGRABLE SYSTEMS RELATED TO CLASSICAL AFFINE FRACTIONAL \mathcal{W} -ALGEBRAS

Assume that $\Lambda_m := -fz^{-m} - pz^{-m-1}$ is a semisimple element in $\hat{\mathfrak{g}}$. Then

$$(6.1) \quad \hat{\mathfrak{g}} = \ker(\text{ad} \Lambda_m) \oplus \text{im}(\text{ad} \Lambda_m).$$

In this case, the following property holds.

Proposition 6.1. *Let $L_m = k\partial + q_m + \Lambda_m \otimes 1$ be a Lax operator defined as in Definition 4.4. Then there exist unique $S \in \bigoplus_{i>0} \hat{\mathfrak{g}}_i \otimes \mathcal{V}_{[d,m]}$ and unique $h(q_m) \in (\ker \text{ad}\Lambda_m \cap \hat{\mathfrak{g}}) \otimes \mathcal{V}_{[d,m]}$ such that*

$$(6.2) \quad L_{m,0} = e^{\text{ad}S}(L_m) = k\partial + \Lambda_m \otimes 1 + h(q_m).$$

Proof. Let $h(q_m) = \sum_{i>-(d+1)m-1} h_i(q_m)$, $q_m = \sum_{i>-(d+1)m-1} q_{m,i}$ and $S = \sum_{i>0} S_i$, where $h_i(q_m), q_{m,i}, S_i \in \hat{\mathfrak{g}}_i \otimes \mathcal{V}_{[d,m]}$. Then equation (6.2) can be rewritten as

$$(6.3) \quad h(q_m) + [\Lambda_m \otimes 1, S] = q_m + [S, k\partial + q_m] + \frac{1}{2}[S, [S, L_m]] + \frac{1}{6}[S, [S, [S, L_m]]] + \cdots.$$

We project the both sides of equation (6.3) onto $\hat{\mathfrak{g}}_{-(d+1)m-\frac{1}{2}} \otimes \mathcal{V}_{[d,m]}$ and get the formula:

$$h_{-(d+1)m-\frac{1}{2}}(q_m) + [\Lambda_m, S_{\frac{1}{2}}] = q_{-(d+1)m-\frac{1}{2}}.$$

By (6.1), we can find $h_{-(d+1)m-\frac{1}{2}}(q_m)$ and $S_{\frac{1}{2}}$ uniquely. Similarly, equating the $(\hat{\mathfrak{g}}_{-(d+1)m} \otimes \mathcal{V}_{[d,m]})$ -part of (6.3), we obtain $h_{-(d+1)m}(q_m)$ and S_1 . Inductively, h_i and S_j for any $i > -(d+1)m-1$ and $j > 0$ are determined uniquely. \square

Since $\Lambda_m z^k \otimes 1$ is in $(\ker(\text{ad}\Lambda_m) \cap \hat{\mathfrak{g}}_{-(d+1)m-1}) \otimes \mathbb{C}$, the space $(\ker(\text{ad}\Lambda_m) \cap \hat{\mathfrak{g}}) \otimes \mathbb{C}$ is nontrivial. Hence we can choose a nonzero element

$$(6.4) \quad b \in (\ker(\text{ad}\Lambda_m) \cap \hat{\mathfrak{g}}) \otimes \mathbb{C}.$$

Let $S \in \mathfrak{n} \otimes \mathcal{V}_{[d,m]}$ be from Proposition 6.1 when L_m is substituted by the universal Lax operator \mathcal{L}_m . Also, let $(e^{-\text{ad}S(x)}b)^{>0}$ and $(e^{-\text{ad}S(x)}b)^{\leq 0}$ be the projections of $e^{-\text{ad}S(x)}b$ onto $\hat{\mathfrak{g}}^{>0} \otimes \mathcal{V}_{[d,m]}$ and $\hat{\mathfrak{g}}^{\leq 0} \otimes \mathcal{V}_{[d,m]}$. Then the following two evolution equations

$$(6.5) \quad \frac{\partial \phi(y)}{\partial t} = \int - \left((e^{-\text{ad}S(x)}b)^{>0} \delta(x-w), \left[\frac{\delta \phi(y)}{\delta u} \delta(y-w), \mathcal{L}_m(w)(q_m) \right] \right)_w dx,$$

$$(6.6) \quad \frac{\partial \phi(y)}{\partial t} = \int \left((e^{-\text{ad}S(x)}b)^{\leq 0} \delta(x-w), \left[\frac{\delta \phi(y)}{\delta u} \delta(y-w), \mathcal{L}_m(w)(q_m) \right] \right)_w dx,$$

where $\phi \in \mathcal{W}(\mathfrak{g}, \Lambda_m, k)$ and $\frac{\delta \phi}{\delta u} = \sum_{(i,j) \in \mathcal{I}} u_i^j \otimes \frac{\delta \phi}{\delta u_i^j}$, are useful to find an integrable system associated to the algebra $\mathcal{W}(\mathfrak{g}, \Lambda_m, k)$.

Proposition 6.2. *Two equations (6.5) and (6.6) are the same evolution equation.*

Proof. We notice that the bilinear form (\cdot, \cdot) on $\hat{\mathfrak{g}} \otimes \mathcal{V}_{[d,m]}$ is $\text{ad}(\hat{\mathfrak{g}} \otimes \mathcal{V}_{[d,m]})$ -invariant, i.e. $(a_1 \otimes f_1, [a_2 \otimes f_2, a_3 \otimes f_3]) = ([a_1 \otimes f_1, a_2 \otimes f_2], a_3 \otimes f_3)$. Also, we have $\int (a \otimes f, [b \otimes g, \partial_x]) dx = -\int (a, b) f \partial_x g dx = \int (b, a) g \partial_x f dx = -\int (b \otimes g, [a \otimes f, \partial_x]) dx$. By subtracting equation (6.5) from (6.6), we obtain

$$(6.6) - (6.5) = - \int \left(\frac{\delta \phi(y)}{\delta u} \delta(y-w), \left[e^{-\text{ad}S(x)} b \delta(x-w), \mathcal{L}_m(w) \right] \right)_w dx.$$

Moreover, since $e^{\text{ad}S(x)} [e^{-S(x)} b \delta(x-w), \mathcal{L}_m(w)] = [b \delta(x-w), e^{\text{ad}S(x)} \mathcal{L}_m(w)]$ and the invariance of the bilinear form holds, we have

$$(6.6) - (6.5) = - \int \left(e^{-\text{ad}S(y)} \frac{\delta \phi(y)}{\delta u} \delta(y-w), \left[b \delta(x-w), e^{\text{ad}S(w)} \mathcal{L}_m(w) \right] \right)_w dx = 0.$$

□

Given b in (6.4), let $H_b : \hat{\mathfrak{g}}_{>-(d+1)m-1}^{\leq 0} \otimes \mathcal{V}_{[d,m]} \rightarrow \mathcal{V}_{[d,m]}$ be a functional defined by

$$(6.7) \quad H_b(q_m(x)) := (b, h(q_m(x))).$$

Then H_b has the following property.

Proposition 6.3. *Let $e^{-adS}b_{<(d+1)m+1}^{\geq 0}$ be the projection of $e^{-adS}b$ onto $\hat{\mathfrak{g}}_{<(d+1)m+1}^{\geq 0}$. Then*

$$\frac{\delta}{\delta u} H_b = e^{-adS}b_{<(d+1)m+1}^{\geq 0}.$$

Proof. For $r \in \hat{\mathfrak{g}}_{>-(d+1)m-1}^{\leq 0} \otimes \mathcal{V}_{[d,m]}$, let $L_m(\epsilon) = k\partial + (q_m(x) + \epsilon r) + \Lambda_m \otimes 1$. By Proposition 6.1, there exist $h(q_m(x) + \epsilon r) \in \ker \text{ad} \Lambda_m \otimes \mathcal{V}_{[d,m]}$ and $S \in \text{im}(\text{ad} \Lambda_m) \otimes \mathcal{V}_{[d,m]}$ satisfying the equation:

$$L_{m,0}(\epsilon) = k\partial + h(q_m(x) + \epsilon r) + \Lambda_m \otimes 1 = e^{\text{ad}S(\epsilon)} L_m(\epsilon).$$

Then

$$(6.8) \quad \frac{d}{d\epsilon} L_{m,0}(\epsilon) = e^{\text{ad}S(\epsilon)} r + \left[\frac{\partial S(\epsilon)}{\partial \epsilon}, e^{\text{ad}S(\epsilon)} L_m(\epsilon) \right] = e^{\text{ad}S(\epsilon)} r + \left[\frac{\partial S(\epsilon)}{\partial \epsilon}, L_{m,0}(\epsilon) \right].$$

Using Taylor expansion and formula (6.8), we have

$$(6.9) \quad \begin{aligned} \left(\frac{\delta}{\delta u} H_b(q_m(x)), r \right) &= \frac{d}{d\epsilon} (b, h(q_m(x) + \epsilon r))|_{\epsilon=0} = \frac{d}{d\epsilon} (b, L_{m,0}(y)(\epsilon))|_{\epsilon=0} \\ &= (e^{-\text{ad}S(x)} b, r) + \left(b, -\partial_x \frac{dS(x)(\epsilon)}{d\epsilon} \Big|_{\epsilon=0} \right). \end{aligned}$$

Applying $\int dx$ to (6.9), we obtain

$$(6.10) \quad \int \left(\frac{\delta}{\delta u} H_b(q_m(x)), r \right) dx = \int (e^{-\text{ad}S(x)} b, r) dx.$$

Since (6.10) holds for any $r \in \hat{\mathfrak{g}}_{>-(d+1)m-1}^{\leq 0} \otimes \mathcal{V}_{[d,m]}$, we conclude that $\frac{\delta H_b}{\delta u} = e^{-\text{ad}S}b_{<(d+1)m+1}^{\geq 0}$. □

By Proposition 6.3, we have

$$\int \left(\frac{\delta H_{z^{-1}b}(q_m(x))}{\delta u} \delta(x-w), zr(y) \right)_w dx = \int ((e^{-\text{ad}S(x)} b)^{>0} \delta(x-w), r(y))_w dx.$$

Hence (6.5) is equivalent to the equation

$$(6.11) \quad \frac{\partial \phi(y)}{\partial t} = \int \{H_{z^{-1}b}(x), \phi(y)\}_1 dx.$$

To write the second equation as a Hamiltonian equation, we denote $\sum_{(i,0) \in \mathcal{I}} u_i^0 \otimes \frac{\delta H_b}{\delta u_i^0}$ by $\frac{\delta H_b}{\delta u^0}$ and $\sum_{(i,j) \in \mathcal{I}, j>0} u_i^j \otimes \frac{\delta H_b}{\delta u_i^j}$ by $\frac{\delta H_b}{\delta u^>}$. Then

$$\begin{aligned}
(6.12) \quad \frac{\partial \phi(y)}{\partial t} &= \int \left(e^{-\text{ad}S(x)} b^{\leq 0} \delta(x-w), \left[\frac{\delta \phi(y)}{\delta u} \delta(y-w), \mathcal{L}_m(w)(q_m) \right] \right)_w dx \\
&= \int \left((e^{-\text{ad}S(x)} b^0 - e^{-\text{ad}S(x)} b^{\geq 0}) \delta(x-w), \left[\frac{\delta \phi(y)}{\delta u} \delta(y-w), \mathcal{L}_m(w)(q_m) \right] \right)_w dx \\
&= \int \left(\frac{\delta H_b(x)}{\delta u^0} \delta(x-w), \left[\frac{\delta \phi(y)}{\delta u} \delta(y-w), \mathcal{L}_m(w)(q_m) \right] \right)_w dx \\
&\quad - \int \left(z^{-1} \frac{\delta H_b(x)}{\delta u^>} \delta(x-w), z \left[\frac{\delta \phi(y)}{\delta u} \delta(y-w), \mathcal{L}_m(w)(q_m) \right] \right)_w dx \\
&= \int \{H_b(x), \phi(y)\}_2 dx.
\end{aligned}$$

By equations (6.11) and (6.12), we get the following theorem.

Theorem 6.4. *Let $b \in (\ker(\text{ad}\Lambda_m) \cap \hat{\mathfrak{g}}_{<0}) \otimes 1$ and $H_b := (b, h(q_m(x)))$. Then*

$$\frac{\partial \phi}{\partial t} = \int \{H_{z^{-1}b}(x), \phi(y)\}_1 dx = \int \{H_b(x), \phi(y)\}_2 dx, \quad \phi \in \mathcal{W}(\mathfrak{g}, \Lambda_m, k).$$

In terms of Poisson vertex algebra theories, Theorem 6.4 can be stated as follows:

Theorem 6.5. *Let b and H_b be defined as in Theorem 6.4 and (6.8). Then*

$$\frac{\partial \phi}{\partial t} = \{H_{z^{-1}b} \lambda \phi\}_1(y)|_{\lambda=0} = \{H_b \lambda \phi\}_2(y)|_{\lambda=0}, \quad \phi \in \mathcal{W}(\mathfrak{g}, \Lambda_m, k).$$

Proof. It suffices to show that this theorem is equivalent to Theorem 6.4. By the results of previous sections, we have

$$\begin{aligned}
(6.13) \quad \int \{H_{z^{-1}b}(x), \phi(y)\}_1 dx &= \sum_{(i,j) \in \mathcal{I}, (p,q) \in \mathcal{I}, n, l \geq 0} \int \frac{\partial H_{z^{-1}b}(x)}{\partial u_i^{j(n)}} \frac{\partial \phi(y)}{\partial u_p^{q(l)}} \partial_x^n \partial_y^l \{u_i^j, u_p^{q+1}\}(y) \delta(x-y) dx \\
&= \sum_{(i,j), (p,q) \in \mathcal{I}, l \geq 0, 0 \leq l_1 \leq l} \left(\partial_y^{l-l_1} \frac{\delta H_{z^{-1}b}(y)}{\delta u_i^j} \right) \frac{\partial \phi(y)}{\partial u_p^{q(l)}} \binom{l}{l_1} \partial_y^{l_1} (\{u_i^j, u_p^{q+1}\}(y)) \\
&= \sum_{(i,j), (p,q) \in \mathcal{I}, l \geq 0} \frac{\partial \phi(y)}{\partial u_p^{q(l)}} \partial_y^l \{u_i^j, u_p^{q+1}\}(y) \frac{\delta H_{z^{-1}b}(y)}{\delta u_i^j} = \{H_{z^{-1}b} \lambda \phi\}_1(y)|_{\lambda=0}.
\end{aligned}$$

Hence $\int \{H_b(x), \phi(y)\}_1 dx = \{H_{z^{-1}b} \lambda \phi\}_1(y)|_{\lambda=0}$. The same procedure works for the second bracket. \square

The following theorem is the main goal of this section.

Theorem 6.6. *Suppose $\frac{\delta H_b}{\delta u^0}$ is not a constant. Then the evolution equation*

$$\frac{\partial \phi}{\partial t} = \{H_{z^{-1}b} \lambda \phi\}_1|_{\lambda=0}$$

is an integrable system. In fact, $H_{z^{-i}b}$, $i \geq 0$, are integrals of motion and they are linearly independent.

Proof. In order to show that each $H_{z^{-i}b}$ is an integral of motion, we need to prove that $\int \{H_{z^{-1}b}, H_{z^{-i}b}\}_1(y) dy = 0$. By Proposition 6.3, we have

$$\begin{aligned} & \int \{H_{z^{-1}b}, H_{z^{-i}b}\}_1(y) dy \\ &= - \iint (e^{-\text{ad}S(x)} z^{-1} b_{<(d+1)m+1}^{\geq 0} \delta(x-w), [e^{-\text{ad}S(y)} z^{-i+1} b_{<(d+1)m+1}^{\geq 0} \delta(y-w), \mathcal{L}_m(w)])_w dx dy \\ &= - \iint (z^{-1} b_{<(d+1)m+1}^{\geq 0} \delta(x-w), [z^{-i+1} b_{<(d+1)m+1}^{\geq 0} \delta(y-w), \mathcal{L}_{m,0}(w)])_w dx dy \\ &= - \iint ([z^{-1} b_{<(d+1)m+1}^{\geq 0} \delta(x-w), z^{-i+1} b_{<(d+1)m+1}^{\geq 0} \delta(y-w)], \mathcal{L}_{m,0}(w))_w dx dy = 0. \end{aligned}$$

Furthermore, Theorem 6.5 implies the independency of the integrals of motion. Indeed, if we denote

$$\{a_{\partial} b\}_{\rightarrow} := \sum_n c_n \partial^n \text{ where } \{a_{\lambda} b\} = \sum_n c_n \lambda^n \in \mathcal{V}_{[d,m]}[\lambda],$$

Theorem 6.5 can be written as

$$\{u_i^j \partial u_p^q\}_1 \rightarrow \frac{\delta H_{z^{-(i+1)}b}}{\delta u_i^j} = \{u_i^j \partial u_p^q\}_2 \rightarrow \frac{\delta H_{z^{-i}b}}{\delta u_i^j},$$

for any $i \geq 0$, $u_i^j, u_p^q \in \mathcal{B}$. Since $\frac{\delta H_b}{\delta u^0}$ is not a constant, there is u_p^0 such that $(u_i^0, u_p^0) \neq 0$. Then $\{u_i^0 \partial u_p^0\}_2 \rightarrow$ increases the total derivative order of $\frac{\delta H_b}{\delta u_i^0}$ by 1. However, $\{u_i^0 \partial u_p^0\}_1 \rightarrow$ preserve the total derivative order of $\frac{\delta H_{z^{-1}b}}{\delta u_i^0}$. Hence H_b and $H_{z^{-1}b}$ are linearly independent. Inductively, each $H_{z^{-i}b}$, for any $i \geq 0$, has a different total derivative order. So $H_{z^{-i}b}$ are linearly independent. \square

Example 6.7. Let $\mathfrak{g} = \mathfrak{sl}_2$ and $m = 1$. Then the associated universal Lax operator is

$$L_1 = k\partial + q_1 + \Lambda_1,$$

where

$$q_1 = \begin{bmatrix} x & f \\ e & -x \end{bmatrix} + \begin{bmatrix} xz & fz \\ 0 & -xz \end{bmatrix} z^{-1} \text{ and } \Lambda_1 = - \begin{bmatrix} 0 & z^{-2} \\ z^{-1} & 0 \end{bmatrix}.$$

There exists a unique linear map $\gamma : \mathfrak{g} \oplus \mathfrak{g}_f z \rightarrow \mathcal{W}(\mathfrak{sl}_2, \Lambda_1, k)$ such that

$$q_1^{can} := \begin{bmatrix} \gamma_x & \gamma_f \\ \gamma_e & -\gamma_x \end{bmatrix} + \begin{bmatrix} 0 & \gamma_{fz} \\ 0 & 0 \end{bmatrix} z^{-1}$$

is gauge equivalent to q_1 . Then $\gamma_x, \gamma_f, \gamma_e, \gamma_{fz}$ freely generate the differential algebra $\mathcal{W}(\mathfrak{g}, \Lambda_1, k)$ (see (5.8)). Let us find $S \in \text{im}(\text{ad}\Lambda_1)$ and $h(q_1^{can}) \in \ker(\Lambda_1) \otimes \mathcal{V}_1$ such that

$$(6.14) \quad k\partial + h(q_1^{can}) + \Lambda_1 \otimes 1 = e^{\text{ad}S} L_1^{can},$$

where $L_1^{can} = k\partial + q_1^{can} + \Lambda_1$. Suppose $h(q_1^{can}) = \sum_{i \geq -1} h_i$, $S = \sum_{i > 0} S_i$ and $h_i, S_i \in \hat{\mathfrak{g}}_i \otimes \mathcal{V}_1$. Then the $(\hat{\mathfrak{g}}_{-1} \otimes \mathcal{V}_1)$ -part of (6.14) is:

$$h_{-1} + [\Lambda_1, S_2] = (q_1^{can})_{-1} = ez^{-1} \otimes \gamma_{fz} + f \otimes \gamma_e.$$

Hence we get

$$(6.15) \quad h_{-1} = \Lambda_1 z \otimes -\frac{1}{2}(\gamma_{fz} + \gamma_e), \quad S_2 = hz \otimes \frac{1}{4}(\gamma_{fz} - \gamma_e).$$

Similarly, by equating $(\hat{\mathbf{g}}_0 \otimes \mathcal{V}_1)$, $(\hat{\mathbf{g}}_1 \otimes \mathcal{V}_1)$, $(\hat{\mathbf{g}}_2 \otimes \mathcal{V}_1)$, $(\hat{\mathbf{g}}_3 \otimes \mathcal{V}_1)$ -parts of (6.14), we have

$$(6.16) \quad \begin{aligned} h_0 &= 0, \quad S_3 = Kz \otimes -\frac{1}{2}\gamma_x, \\ h_1 &= \Lambda_1 z^2 \otimes \left(-\frac{1}{2}\gamma_f - \frac{1}{4}(\gamma_{fz} - \gamma_e)^2\right), \quad S_4 = hz^2 \otimes \left(\frac{1}{4}\gamma_f + \frac{1}{16}\gamma_{fz}^2 - \frac{3}{16}\gamma_e^2 + \frac{1}{8}\gamma_{fz}\gamma_e\right), \\ h_2 &= 0, \\ h_3 &= \Lambda_1 z^3 \otimes \left(\frac{1}{2}\gamma_x^2 + (\gamma_e - \gamma_{fz})\left(\frac{1}{16}\gamma_{fz}^2 - \frac{1}{16}\gamma_e^2 + \frac{1}{4}\gamma_f\right)\right), \end{aligned}$$

where $K = (-e + fz)$.

Let $bz^{-n} := -\frac{1}{2}(e + fz)z^{-n} \otimes 1$ and let $H_n(q_1) := H_{bz^{-n}}(q_1) = (bz^{-n}, h(q_1))$. Then we obtain

$$H_0 = -\frac{1}{2}\gamma_f - \frac{1}{4}(\gamma_{fz} - \gamma_e)^2, \quad H_1 = \frac{1}{2}\gamma_x^2 + (\gamma_e - \gamma_{fz})\left(\frac{1}{16}\gamma_{fz}^2 - \frac{1}{16}\gamma_e^2 + \frac{1}{4}\gamma_f\right).$$

Using formulas (5.9)-(5.12), we obtain the following Poisson λ -brackets:

$$(6.17) \quad \begin{aligned} \{H_{0\lambda}\gamma_f\}_1 &= k\lambda, \\ \{H_{0\lambda}\gamma_x\}_1 &= \{H_{0\lambda}\gamma_e\}_1 = \{H_{0\lambda}\gamma_{fz}\}_1 = 0, \end{aligned}$$

$$(6.18) \quad \begin{aligned} \{H_{0\lambda}\gamma_f\}_2 &= \{H_{1\lambda}\gamma_f\}_1 = \left(\gamma_x + \frac{1}{2}k(\lambda + \partial)\right)(\gamma_e - \gamma_{fz}), \\ \{H_{0\lambda}\gamma_x\}_2 &= \{H_{1\lambda}\gamma_x\}_1 = -\frac{1}{2}\gamma_e(\gamma_{fz} - \gamma_e) - \frac{1}{2}\gamma_f, \\ \{H_{0\lambda}\gamma_e\}_2 &= \gamma_x - \frac{1}{2}k\lambda, \quad \{H_{1\lambda}\gamma_e\}_1 = \gamma_x, \\ \{H_{0\lambda}\gamma_{fz}\}_2 &= -\gamma_x + \frac{1}{2}k\lambda, \quad \{H_{1\lambda}\gamma_{fz}\}_1 = -\gamma_x, \end{aligned}$$

$$(6.19) \quad \begin{aligned} \{H_{1\lambda}\gamma_f\}_2 &= \left(\frac{1}{2}\gamma_x + \frac{1}{4}k(\lambda + \partial)\right)(\gamma_{fz}^2 - \gamma_e^2) + \frac{1}{2}k(\lambda + \partial)\gamma_f, \\ \{H_{1\lambda}\gamma_x\}_2 &= \left(\frac{1}{4}\gamma_e(\gamma_e - \gamma_{fz}) - \frac{1}{4}f\right)(\gamma_e + \gamma_{fz}), \\ \{H_{1\lambda}\gamma_e\}_2 &= \gamma_e\gamma_x + \left(-\frac{1}{2}\gamma_x + \frac{1}{4}k(\lambda + \partial)\right)(\gamma_e - \gamma_{fz}), \\ \{H_{1\lambda}\gamma_{fz}\}_2 &= -\gamma_e\gamma_x - \left(-\frac{1}{2}\gamma_x + \frac{1}{4}k(\lambda + \partial)\right)(\gamma_e - \gamma_{fz}). \end{aligned}$$

As a consequence, two equations

$$(6.20) \quad \begin{aligned} \frac{d\gamma}{dt} &= \{H_{0\lambda}\gamma\}_2|_{\lambda=0} = \{H_{1\lambda}\gamma\}_1|_{\lambda=0}, \\ \frac{d\gamma}{dt} &= \{H_{1\lambda}\gamma\}_2|_{\lambda=0}, \end{aligned}$$

are compatible integrable systems. By (6.18), the first equation in (6.20) is as follows:

$$(6.21) \quad \begin{cases} \frac{d\gamma_f}{dt} = (\gamma_x + \frac{1}{2}k\partial)(\gamma_{fz} - \gamma_e) \\ \frac{d\gamma_x}{dt} = -\frac{1}{2}\gamma_e(\gamma_{fz} - \gamma_e) - \frac{1}{2}\gamma_f \\ \frac{d\gamma_e}{dt} = -\frac{d\gamma_{fz}}{dt} = \gamma_x. \end{cases}$$

Since $\gamma_e + \gamma_{fz}$ is in the center of $\mathcal{W}(\mathfrak{g}, f, k)$, we take quotient by the center of $\mathcal{W}(\mathfrak{g}, f, k)$ and obtain the following equation:

$$(6.22) \quad \begin{cases} \frac{d\gamma_f}{dt} = -2(\gamma_x + \frac{1}{2}k\partial)(\gamma_e) \\ \frac{d\gamma_x}{dt} = \gamma_e^2 - \frac{1}{2}\gamma_f \\ \frac{d\gamma_e}{dt} = \gamma_x. \end{cases}$$

Eliminating γ_f and γ_x , we get the equation

$$(6.23) \quad (\gamma_e)_{ttt} = 3\gamma_e(\gamma_e)_t + \frac{1}{2}k(\gamma_e)_x.$$

This equation is the KdV equation with x and t exchanged.

REFERENCES

- [1] N.J. Burroughs, M.F. De Groot, T.J. Hollowood, J.L.Miramontes, Generalized Drinfel'd-Sokolov hierarchies. Commun.Math.Phys. 153, (1993).
- [2] A. Barakat, A. De Sole, V.G. Kac, Poisson vertex algebras in the theory of Hamiltonian equations. Jpn. J. Math. 4 (2009), no. 2, 141-252
- [3] M.F. De Groot, T.J. Hollowood, J.L.Miramontes, Generalized Drinfel'd-Sokolov hierarchies. Commun.Math.Phys. 145, (1992).
- [4] A. De Sole, V. G. Kac, Finite vs affine W -algebras, Jpn. J. Math. 1 (2006), no. 1, 137-261
- [5] A. De Sole, V. G. Kac, D. Valeri, Classical W -algebras and generalized Drinfeld-Sokolov bi-Hamiltonian systems within the theory of Poisson vertex algebras, arXiv:1207.6286
- [6] V.A. Fateev, S.L. Lukyanov, The models of two dimensional conformal quantum field theory with \mathbb{Z}_n symmetry, Int. J. Mod. Phys. A 3 (1988), 507-520.
- [7] L.D. Faddeev, L.A. Takhtajan, Hamiltonian approach in soliton theory, Nauka, 1986
- [8] W. L. Gan, V. Ginzburg, Quantization of Slodowy slices, Int. Math. Res. Not. 2002, no. 5, 243-255
- [9] Victor G. Kac, Alexei Rudakov, Representations of the exceptional Lie superalgebra $E(3,6)$. III. Classification of singular vectors. J. Algebra Appl. 4 (2005), no. 1, 15-57
- [10] Victor G. Kac, Minoru Wakimoto, Quantum reduction and representation theory of superconformal algebras. Adv. Math. 185 (2004), no. 2, 400-458.
- [11] A. Premet, Special transverse slices and their enveloping algebras, Adv. Math. 170 (2002), no. 1, 1-55

- [12] A. Premet, Enveloping algebras of Slodowy slices and the Joseph ideals, J. Eur. Math. Soc. (JEMS) 9 (2007), no. 3, 487-543
- [13] V.G. Drinfel'd, V.V. Sokolov, Lie algebras and equations of Korteweg-de Vries Type. J.Sov.Math.Dokl. 23, 1975 (1985)
- [14] V.G. Kac, Infinite dimensional Lie algebras. 2nd ed. Cambridge University Press 1985
- [15] V.G. Kac, D.H. Peterson, 112 Constructions of the basic representation of the loop group of E_8 , Symposium on Anomalies, geometry and topology,
- [16] Suh, Ph.D. Thesis, Structure of classical \mathcal{W} -algebras (2013)
- [17] Charles A. Weibel, An introduction to homological algebra. Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994.
- [18] G.Wilson, The modified Lax and two-dimensional Toda Lattice equations associated with simple Lie algebras. Ergod.Th. and Dynam.Sys. 1, 361 (1981)